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K_2 AND K_3 OF THE CIRCLE

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Let k be $\mathbb{Z}[\frac{1}{2}]$, $\mathbb{Q}$ or $\mathbb{R}$, and set $A=k[x,y]/(x^2+y^2-1)$. We compute $K_2(A)$ and $K_3(A)$. Our method is to construct a map $\varphi: K_*(k[\mathrm{i}])\to K_{*+1}(A)$ and compare this to a localization sequence.

We give three applications. We show that φ accounts for the primitive elements in $K_2(A)$, and compare our results to computations of Bloch [1] for group schemes. Secondly, we consider the problem of basepoint independence, and indicate the interplay of geometry upon the K-theory of affine schemes obtained by glueing points of Spec(A). Third, we can iterate the construction to compute the K-theory of the torus ring $A\otimes_k A$.

Introduction

Let k be a field of characteristic $\neq 2$ and set $A=k[x,y]/(x^2+y^2-1)$, the coordinate ring of the 'circle scheme'. Roberts [15] has computed $K_*(A)$ up to extension, in terms of the K-theory of k and $k[\mathrm{i}]$. The purpose of this paper is primarily to solve this extension problem in terms of a map $\varphi: K_*(k[\mathrm{i}])\to K_{*+1}(A)$.

The construction we use generalizes to the following situation: k is a regular ring containing b, c, and d^{-1}, $d=4c-b^2$. The ring A is replaced by $k[x,y]/(x^2+bxy+cy^2-d)$, and $k[\mathrm{i}]$ is replaced by $l=k[\alpha]$, $\alpha^2+b\alpha+c=0$. In Section 0 we give the necessary ring-theoretic background for this generalization and show that Spec(A) is a commutative group scheme over k.

In Section 1 we construct φ and give its main properties. We obtain the general result when k, l are fields: $K_2(A)=K_2(k)\oplus\varphi(l^*)\oplus E$, where E is an elementary 2-group. In Section 2 we give examples and computations. In particular, we compute

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the (lower) K-groups of the torus scheme.

In Section 3 we describe the primitive elements of $K_*(A)$. When k, l are fields, $\varphi(l^*)$ is the subgroup of primitive elements of $K_2(A)$.

In Section 4 we apply the computations of Section 2 to analyze the manner in which the kernel of $p^*: K_*A \to K_*k$ varies with the choice of a section $p: A \to k$. For the circle scheme over $\mathbb{R}$, this $\ker(p^*)$ is the same for points of S^1 differing by a root of unity, but different if they differ by a transcendental element of $S^1 \subset \mathbb{C}$.

If Y is an affine scheme obtained from $\operatorname{Spec}(A)$ by glueing k-rational points $p_1, \dots, p_l$ together, the K-theory of Y depends upon the way $\ker(p_i^*)$ varies with i. We illustrate this dependence for $k = \mathbb{R}$ in the two cases $l = \mathbb{R} \times \mathbb{R}$ ($A = \mathbb{R}[t, t^{-1}]$) and $l = \mathbb{C}$ ($A = \mathbb{A}[x, y]/(x^2 + y^2 - 1)$). In the latter case, glueing $y = \pm 1$ gives a figure eight: $Y(\mathbb{R}) \simeq S^1 \vee S^1$. The topological maps $\tilde{K}_0(Y) \to \widetilde{KO}(S^1 \vee S^1) = (\mathbb{Z}/2)^2$, $\mathrm{SK}_1(Y) \to [S^1 \vee S^1, SO] = (\mathbb{Z}/2)^2$ are split epis, whose kernels are uncountable $\mathbb{Q}$-vector spaces.

0. Preliminaries

In this section we set up the ring-theoretic background we will need.

Standing Assumption. k is a regular, commutative (noetherian) ring, and $\lambda: k \to l$ is an étale map of commutative rings with l free of rank 2 over k.

Remark. If we merely assume that l is free of rank 2 over k with basis $\{u, v\}$, write $1 = a_1 u + b_1 v$, $u^2 = a_2 u + b_2 v$, $uv = a_3 u + b_3 v$, and substitute into $u = a_1 u^2 + b_1 uv$, we obtain $1 = a_1 a_2 + b_1 a_3$. That is, the vector (a_1, b_1) is unimodular, and 1 is part of some basis of l over k. If we take $\alpha \in l$ such that $(1, \alpha)$ is a basis of l over k, then $l = k[\alpha] = k[t]/(f)$, where α is the image of t in l and $f(t) = t^2 + bt + c = (t - \alpha)(t - \bar{\alpha})$. Let $d = 4c - b^2 = -(\alpha - \bar{\alpha})^2$. Then the following are equivalent:

(a) d is a unit of k,
(b) l is smooth over k,
(c) l is unramified over k,
(d) l is étale over k.

Write $F(T_0, T_1) = T_0^2 + bT_0T_1 + cT_1^2$ with b and c as above, and set

$$A = k[x, y]/(F(x, y) - d).$$

Note that $F(b, -2) = d$, so that there is a section $\varepsilon_0: A \to k$ defined by $\varepsilon_0(x) = b$, $\varepsilon_0(y) = -2$. Here are some examples.

Examples 0.1. (a) The generic example is the case $k_0 = \mathbb{Z}[b, c, (4c - b^2)^{-1}]$, $f(t) = t^2 + bt + c$. There is always a map $k_0 \to k$ inducing $A_0 \to A$, $l_0 \to l$.

(b) The 'circle scheme' arises if k is a regular ring containing $\frac{1}{2}$ and $f(t) = t^2 + 1$ ($d = 4$). For then $l = k[\mathrm{i}]$,

$$A = k[X, Y]/(X^2 + Y^2 - 1),$$

$$\varepsilon(X) = 1, \qquad \varepsilon(Y) = 0,$$

where $X = -y/2$, $Y = x/2$.

(c) When k is regular of characteristic 2, take $f(t) = t^2 + t + c$. We have $A = k[x, y]/(x^2 + xy + cy^2 + 1)$, $\varepsilon(x) = 1$ and $\varepsilon(y) = 0$. This is the general case if $2 = 0$: if l is presented with $b \neq 1$, dividing α, t and x by b puts f, F and A in the described form.

Observation. The ring A depends only on k and l, not on the form F used to define it. For if we present $l = k[\alpha']$ we have $\alpha = a_0\alpha' + a_1$, a_0 a unit in k, $a_1 \in k$. The resulting ring is

$$A' = k[x', y']/(F(a_0x' + a_1y', y') - d).$$

The linear change of variables $x = a_0x' + a_1y'$, $y = y'$ gives the isomorphism of A' with A.

Our study of the K-theory of A will be based on the following geometrical interpretation. The equation $F(T_0, T_1) = 0$ defines a closed immersion

$$j: \mathrm{Spec}(l) \to \mathbb{P}^1_k.$$

The complement is $\mathrm{Spec}(B)$, where

$$\begin{aligned} B &= (\text{degree 0 part of } k[T_0, T_1]_F) \cong k[T_0^2, T_0T_1, T_1^2]/(F(T_0, T_1) - 1) \\ &\cong k(X, Y, Z)/(X + bY + cZ - 1, XZ - Y^2) \\ &\cong k[Y, Z]/(Y^2 + bYZ + cZ^2 - Z) \\ &= k[x, y]/(x^2 + bxy + cy^2 - d) = A, \end{aligned}$$

for $x = dY + b$, $y = dZ - 2$. Under the isomorphism (degree 0 part of $k[T_0, T_1]_F) \cong k[Y, Z]/(Y^2 + bYZ + cZ^2 - Z)$ we have $Y = T_0T_1/F$ and $Z = T_1^2/F$.

We now consider extension of scalars from k to l. All tensor products will be over k. First of all, $l \otimes l = l[t]/(f) = l[t]/(t - \alpha)(t - \bar{\alpha}) \cong l \times l$, where the last isomorphism is given by $t = \alpha$, $t = \bar{\alpha}$ respectively. We have $\mathrm{Spec}(l) \subset \mathbb{P}^1_k \supset \mathrm{Spec}(A)$. This extends to $\mathrm{Spec}(l \times l) \subset \mathbb{P}^1_l \supset \mathrm{Spec}(A \otimes l)$, where the closed subscheme $\mathrm{Spec}(l \times l)$ is again given by $0 = T_0^2 + bT_0T_1 + cT_1^2 = (T_0 - \alpha T_1)(T_0 - \bar{\alpha}T_1)$. Since $\alpha - \bar{\alpha}$ is a unit of l, we can introduce $S_0 = T_0 - \alpha T_1$, $S_1 = T_0 - \bar{\alpha}T_1$ as a new set of homogeneous co-ordinates for $\mathbb{P}^1_l$. Let $u = S_0/S_1$. The complement of $\mathrm{Spec}(l \times l)$ in $\mathbb{P}^1_l$ is then $\mathrm{Spec}(l[u, u^{-1}])$, so $A \otimes l = l[u, u^{-1}]$. One can check that $u = (\alpha - \bar{\alpha})^{-1}(x - \alpha y)$ and $u^{-1} = (\bar{\alpha} - \alpha)^{-1}(x - \bar{\alpha}y)$.

Let σ be the k-algebra automorphism of l defined by $\sigma(\alpha) = \bar{\alpha}$. This induces automorphisms of $A \otimes l$, $l[T_0, T_1]$, and $\mathbb{P}^1_l = \mathbb{P}^1_k \times \mathrm{Spec}(l)$ which will also be denoted σ. The fixed rings of σ are k, A, $k[T_0, T_1]$ respectively. Since $\sigma(S_0) = S_1$ and $\sigma(S_1) = S_0$ we have $\sigma(u) = u^{-1}$, and the two copies of $\mathrm{Spec}(l) \subset \mathbb{P}^1_l$ given respectively by $S_0 = 0$, $S_1 = 0$ are interchanged by σ. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(l)\times\mathrm{Spec}(l) & \xrightarrow[S_0=0,\,S_1=0]{} & \mathbb{P}^1_l & \longleftarrow & \mathrm{Spec}(A\otimes l) \\ \downarrow{\scriptstyle (1,\sigma)} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda^A} \\ \mathrm{Spec}(l) & \xrightarrow[j]{} & \mathbb{P}^1_k & \longleftarrow & \mathrm{Spec}(A) \end{array} \tag{0.2}$$

We note that A is regular since Spec(A) is an open subscheme of $\mathbb{P}^1_k$. The ring l is regular by [4, IV (6.5.2)(ii)], using the definition of smooth in [4, IV (6.8.1)].

0.3. We now describe the sections (augmentations) $\varepsilon: A\to k$. First we show that sections are in 1–1 correspondence with the units μ of l satisfying $\sigma(\mu)=\mu^{-1}$. To a given section $\varepsilon: A\to k$, we associate the element $\mu=(\varepsilon\otimes l)(u)$ of l, where $\varepsilon\otimes l: A\otimes l=l[u,u^{-1}]\to l$. Note that μ is a unit and that $\sigma(\mu)=\sigma(\varepsilon\otimes l)(u)=(\varepsilon\otimes l)(\sigma(u))=(\varepsilon\otimes l)(u^{-1})=\mu^{-1}$. Conversely, given such a unit μ of l, we consider the l-algebra homomorphism $\delta: A\otimes l=l[u,u^{-1}]\to l$ defined by $\delta(u)=\mu$. Since $\sigma(\mu)=\mu^{-1}$, δ induces a map of σ-invariant rings $\varepsilon: A=(A\otimes l)^\sigma\to l^\sigma=k$ for which $\varepsilon\otimes l=\delta$. The construction of ε from μ completes the correspondence.

The standard section ε_0 corresponds to the unit 1. To the unit -1 corresponds the section defined by $x\mapsto -b$, $y\mapsto 2$. These are the only two generic sections, since ± 1 are the only units of l_0 satisfying $\sigma(\mu)=\mu^{-1}$. To see this, note that $k_0=\mathbb{Z}[b,c,(4c-b^2)^{-1}]$, $l_0=\mathbb{Z}[b+2\alpha,(b+2\alpha)^{-1},\alpha]$, and $\sigma(b+2\alpha)=-(b+2\alpha)$.

Now let $\delta,\varepsilon: A\to k$ be any two sections, corresponding to the units μ, ν of l, respectively. Consider the l-algebra automorphism τ of $l[u,u^{-1}]$ defined by $\tau(u)=(\mu^{-1}\nu)u$. Note that $\tau\sigma=\sigma\tau$, so that τ induces a k-algebra automorphism of the fixed subring $A=(l[u,u^{-1}])^\sigma$. Since $\varepsilon\otimes l=(\delta\otimes l)\tau$, we also have $\varepsilon=\delta\tau$. Thus the two augmentations of A differ only by a k-algebra automorphism τ of A.

The automorphism τ of A is in fact a linear change of variables. To see this, set $x'=\tau(x)$, $y'=\tau(y)$. Then $x'-\alpha y'=(\alpha-\bar\alpha)\tau(u)=(\alpha-\bar\alpha)(\mu^{-1}\nu)u=(\mu^{-1}\nu)(x-\alpha y)$. Note that $\mu^{-1}\nu=[(\delta\otimes l)u]^{-1}[(\varepsilon\otimes l)u]=d^{-1}[\delta(x)-\bar\alpha\delta(y)][\varepsilon(x)-\alpha\varepsilon(y)]$. If we write out the matrices of $\varepsilon(x)-\alpha\varepsilon(y)$, $\delta(x)-\bar\alpha\delta(y)$ relative to the basis $(1,-\alpha)$ of $A\otimes l$, we obtain the desired linear relation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = d^{-1}\begin{bmatrix} \varepsilon(x) & -c\varepsilon(y) \\ \varepsilon(y) & \varepsilon(x+by) \end{bmatrix}\begin{bmatrix} \delta(x+by) & c\delta(y) \\ -\delta(y) & \delta(x) \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

0.4. We conclude this section by showing that A is a (cocommutative) Hopf algebra over k, or equivalently, that Spec(A) is a commutative group scheme over Spec(k). Now $l[u,u^{-1}]=A\otimes l$ is a Hopf algebra over l with counit $u\mapsto 1$, coproduct $\Delta(u)=u\otimes u$, and antipode $S(u)=u^{-1}$. It follows from [18] that this structure is induced by a Hopf algebra structure on A by base change.

In this case, the counit on A is our assumed ε, while the coproduct and antipode are given by:

$$d\Delta(x)=\varepsilon(x+by)x\otimes x-c\varepsilon(x)y\otimes y+c\varepsilon(y)(x\otimes y+y\otimes x),$$

$$d\Delta(y) = -\varepsilon(y)x\otimes x + \varepsilon(bx+cy)y\otimes y + \varepsilon(x)(x\otimes y + y\otimes x),$$

$$dS(x) = \varepsilon(x^2 - cy^2)x + \varepsilon(bx^2 + 2cxy)y,$$

$$dS(y) = \varepsilon(2xy + by^2)x - \varepsilon(x^2 - cy^2)y.$$

Remark. Suppose $d=1$. (If d is a square in k, replace $f(t)=F(t,1)$ by $F(t,d^{-\frac{1}{2}})$, α by $d^{\frac{1}{2}}\alpha$ to obtain $d=1$.) We can then choose $\varepsilon(x)=1$, $\varepsilon(y)=0$. This has been done in Examples 0.1 (b) and (c). The group scheme Spec(A) is then given by

$$\Delta(x) = x\otimes x - cy\otimes y, \qquad \Delta(y) = y\otimes x + x\otimes y + by\otimes y,$$

$$S(x) = x + by \qquad S(y) = -y.$$

1. $K_*(A)$

The purpose of this section is to establish the basic properties of the map $\varphi : K_{*-1}(l) \to K_*(A)$. The definition of φ is due to D. Grayson.

We must first discuss the exact sequence of localization. The discussion in 0.3 shows that there is no loss of generality in using only the standard section $\varepsilon_0 : A \to k$. We will do this, dropping the subscript 0 for convenience of notation. Thus we now have $(\varepsilon\otimes l)(u)=1$. Let $\tilde{K}_n(A)$ denote the kernel of $\varepsilon^* : K_n(A) \to K_n(k)$, so that $K_n(A) = K_n(k) \oplus \tilde{K}_n(A)$. The exact sequence of localization is

$$\cdots K_{n+1}(A) \xrightarrow{\partial} K_n(l) \xrightarrow{j_*} K_n(\mathbb{P}^1_k) \to \cdots \to K_0(A) \to 0.$$

Now $K_n(\mathbb{P}^1_k) = K_n(k)[\mathfrak{o}] \oplus K_n(k)[\mathfrak{o}_p]$, where $\mathfrak{o}$ is the structure sheaf of $\mathbb{P}^1_k$ and $\mathfrak{o}_p$ is the structure sheaf of the closed subscheme Spec(k) $\subset \mathbb{P}^1_k$ induced by ε. There is an exact sequence

$$0 \to \mathfrak{o}(-1) \to \mathfrak{o} \to \mathfrak{o}_p \to 0,$$

so in $K_0(\mathbb{P}^1_k)$ we have $[\mathfrak{o}_p] = [\mathfrak{o}] - [\mathfrak{o}(-1)]$. Furthermore, $\mathfrak{o}(-1)$ restricted to Spec(A) is ker(ε), so ker(ε) is an invertible ideal of Spec(A) satisfying $(\ker(\varepsilon))^2 = A$. Also $\mathfrak{o}_p$ restricts to $\varepsilon_*(1) = \eta$, which in $K_0(A)$ equals $1 - [\ker(\varepsilon)]$. As in [15] the map j_* factors as

$$K_n(l) \xrightarrow{\lambda_*} K_n(k) \xrightarrow{[\mathfrak{o}_p]} K_n(\mathbb{P}^1_k).$$

The projection formula for $x \in K_*(A)$ gives

$$x\cdot\eta = x\cdot\varepsilon_*(1) = \varepsilon_*(\varepsilon^*(x)\cdot 1) = \varepsilon_*\varepsilon^*(x).$$

In particular, for $x \in K_*(k) \subseteq K_*(A)$ we have $x\cdot\eta = \varepsilon_*(x)$. Finally, we have $\varepsilon^*\varepsilon_*(x) = \varepsilon^*(x)\cdot\varepsilon^*(\eta) = \varepsilon^*(x)\cdot 0 = 0$ for $x \in K_n(k)$, so ε_* maps $K_n(k)$ into $\tilde{K}_n(A)$. In summary:

Theorem 1.1. *There is a long exact sequence*

$$\cdots K_n(l) \xrightarrow{\lambda_*} K_n(k) \xrightarrow{\varepsilon_*} \tilde{K}_n(A) \xrightarrow{\partial} K_{n-1}(l) \xrightarrow{\lambda_*} K_{n-1}(k) \cdots.$$

This sequence is in [15], but the homomorphism $K_n(k)\to\tilde{K}_n(A)$ there was not explicitly identified as ε_*.

Corollary 1.2. *We have exact sequences*

$$0\to K_n(k)/\lambda_*K_n(l)\xrightarrow{\varepsilon_*}\tilde{K}_n(A)\to\operatorname{Ker}(K_{n-1}(l)\xrightarrow{\lambda_*}K_{n-1}(k))\to 0.$$

In particular, $K_0(A)=K_0(k)\oplus(K_0k/\lambda_*K_0l)\cdot\eta$, *where* $\eta=\varepsilon_*(1)$.

The structure of $K_0(A)$ as a $K_0(k)$-algebra is given by Corollary 1.2 and the relation $\eta^2=0$. This can be seen as follows: as a $K_0(k)$-algebra, $K_0(\mathbb{P}^1_k)=K_0(k)[X]/(X^2)$, where $X=[\mathfrak{o}]-[\mathfrak{o}(-1)]$. The restriction homomorphism $K_0(\mathbb{P}^1_k)\to K_0(A)$ is a $K_0(k)$-algebra surjection, and X restricts to η. Explicitly, this says that $\lambda_*K_0(l)$ is an ideal of $K_0(k)$ – this may be seen directly from the projection formula – and that the ring structure on $K_0(A)$ is

$$(a+\bar{b}\eta)(c+\bar{d}\eta)=ac+(\overline{ad+bc})\eta.$$

Here $a,b,c,d\in K_0(k)$ and the $^-$ denotes the class of an element in the $K_0(k)$-module $K_0(k)/\lambda_*K_0(l)$.

Remark. The exact sequences in Theorem 1.1 and Corollary 1.2 are natural in k, i.e., for each regular k-algebra k' there is a chain map from the sequences for k, l, A, etc. to the sequences for k', $l\otimes k'$, $A\otimes k'$, etc. To see this, note that by Lemma 1.3 below the following diagram commutes:

$$\begin{tikzcd}
K_*(l) \arrow[r, "\lambda_*"] \arrow[d] & K_*(k) \arrow[r, "{[\mathfrak{o}_p]}"] \arrow[d] & K_*(\mathbb{P}^1_k) \arrow[r] \arrow[d] & K_*(A) \arrow[d] \\
K_*(l\otimes k') \arrow[r, "\lambda'_*"] & K_*(k') \arrow[r, "{[\mathfrak{o}'_p]}"] & K_*(\mathbb{P}^1_{k'}) \arrow[r] & K_*(A\otimes k').
\end{tikzcd}$$

Thus the argument of [13, Remark 3.4, p. 128] applies: the map $\otimes\, k'$ induces a map of fibrations, hence a map of the resultant long exact homotopy sequences.

Lemma 1.3. *In the diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "B\otimes g"] \\
C \arrow[r, "f\otimes C"] & B\otimes_A C,
\end{tikzcd}$$

suppose that B has a finite resolution by finitely generated projective A-modules, and that B and C are Tor*-independent over A, i.e.,* $\operatorname{Tor}_i^A(B,C)=0$ *for $i>0$. Then*

$$g^*f_*=(f\otimes C)_*(B\otimes g)^*:K_*(B)\to K_*(C).$$

Proof. This is essentially Proposition 2.11 on p. 127 of [13]: if $P_.$ is an A-projective resolution of B, then $P_. \otimes C$ is an exact C-projective resolution of $B \otimes C$, so $(f \otimes C)_*$ exists. If $\mathbf{L}$ denotes the finite homological dimension A-modules which are Tor-independent from C, then f_* maps $\mathbf{P}(B)$ into $\mathbf{L}$, and '$\otimes C$' $: \mathbf{L} \to \mathbf{H}(C)$ is exact, so $g^* f_*$ is induced by '$\otimes_A C$' $: \mathbf{P}(B) \to \mathbf{H}(C)$. This map factors as $\mathbf{P}(B) \to \mathbf{P}(B \otimes C) \to \mathbf{H}(C)$, which induces the map $(f \otimes C)_* (B \otimes g)^*$ on K-theory. □

Our intention is to solve the extension problem posed by Corollary 1.2. Loday [10] has defined a product $K_i(R) \times K_j(R) \to K_{i+j}(R)$ for any commutative ring R. This product will be written either in symbol notation $\{\ ,\ \}$ or simply by a dot. We define a homomorphism

$$\varphi : K_{*-1}(l) \to K_*(A)$$

by $\varphi(\zeta) = \lambda_*^A \{\zeta, u\}$, where $\lambda_*^A : K_*(A \otimes l) \to K_*(A)$ is the transfer. (ζ is in the subgroup $K_{*-1}(l)$ of $K_{*-1}(A \otimes l)$.)

The map φ is natural in k. To see this, consider a map $\mu : k \to k'$ and an element $\zeta \in K_{*-1}(l)$. By abuse, we will also use μ to denote the maps $l \to l \otimes k'$, $A \to A \otimes k'$, and $A \otimes l \to A \otimes l \otimes k'$. By Lemma 1.3, with $B = A \otimes l$ and $C = A \otimes k'$, we have:

$$\mu^* \varphi(\zeta) = \mu^* \lambda_*^A \{\zeta, u\} = (\lambda^A \otimes k')_* \{\mu^*(\zeta), u\} = \varphi(\mu^*(\zeta)).$$

Corollary 1.4. *$\varepsilon^* \varphi = 0$, so the image of φ lies in $\tilde{K}_*(A)$.*

Proof. We have a diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda^A} & A \otimes l \\ \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon \otimes l} \\ k & \xrightarrow{\lambda} & l. \end{array}$$

From the above lemma we obtain $\varepsilon^* \varphi(\zeta) = \varepsilon^* \lambda_*^A \{\zeta, u\} = \lambda_* (\varepsilon \otimes l)^* \{\zeta, u\}$. Now $A \otimes l = l[u, u^{-1}]$ and $(\varepsilon \otimes l)(u) = 1$, while $\zeta \in K_*(l)$. By naturality of the product [10, Theorem 2.1.11],

$$(\varepsilon \otimes l)^* \{\zeta, u\} = \{\zeta, (\varepsilon \otimes l)^* u\} = \{\zeta, 1\} = 0. \qquad \square$$

Proposition 1.5. *The composition $\partial\varphi : K_n(l) \to K_n(l)$ is $\zeta \mapsto \zeta - \bar{\zeta}$, where $\bar{\zeta} = \sigma(\zeta)$.*

Proof. We use (0.2) to compare the two localization exact sequences via the transfer map:

$$\begin{array}{ccccc} K_{n+1}(\mathbb{P}_l^1) & \longrightarrow & K_{n+1}(l[u, u^{-1}]) & \xrightarrow{\partial'} & K_n(l_0) \times K_n(l_\infty) \\ \Big\downarrow{\scriptstyle \lambda_*} & & \Big\downarrow{\scriptstyle \lambda_*^A} & & \Big\downarrow{\scriptstyle (1, \sigma)} \\ K_{n+1}(\mathbb{P}_k^1) & \longrightarrow & K_{n+1}(A) & \xrightarrow{\partial} & K_n(l). \end{array}$$

If we examine the proof of the Fundamental Theorem [6, p. 236], we see that for $\zeta \in K_n(l)$ we have $\partial'\{\zeta, u\} = (\zeta, ?)$ and similarly $\partial'\{\zeta, u^{-1}\} = (?, \zeta)$. This gives $\partial'\{\zeta, u\} = (\zeta, -\zeta)$. Finally, we have $\partial\varphi(\zeta) = \partial\lambda_*^A\{\zeta, u\} = (1, \sigma)(\zeta, -\zeta) = \zeta - \bar{\zeta}$. □

We now note that the projection formula holds for Loday's product [10], applied to $\lambda^A : A \to A \otimes l$.

Proposition 1.6. *Let $i : A \to B$ be a ring map, such that B is a finitely generated projective A-module. Then the projection formula holds for Loday's product, that is: if $x \in K_p(A)$, $y \in K_{n-p}(B)$, then $x \cdot i_* y = i_*(i^* x \cdot y)$ in $K_n(A)$.*

Proof. This is Theorem 4.1 on p. 202 of [2]. □

We now calculate K_0 and K_1 of the generic rings k_0 and l_0. This will be used in proving Theorem 1.9, as well as in Section 2. Recall that $k_0 = \mathbb{Z}[b, c, (4c - b^2)^{-1}]$, and $l_0 = k_0[\alpha] = \mathbb{Z}[b + 2\alpha, (b + 2\alpha)^{-1}, \alpha]$. We have $K_0(k_0) = K_0(l_0) = \mathbb{Z}$, because k_0 and l_0 are localizations of regular rings whose K_0 is $\mathbb{Z}$. Also, $\mathrm{SK}_1(l_0) = 0$ and $K_1(l_0) = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ on units $\pm 1, b + 2\alpha$. To compute $K_1(k_0)$, we use the localization exact sequence, whose relevant portion is

$$K_2(k_0[\tfrac{1}{2}]) \xrightarrow{\partial} K_1(k_0/(2)) \xrightarrow{\pi_*} K_1(k_0) \longrightarrow K_1(k_0[\tfrac{1}{2}]).$$

Since $k_0/(2) = \mathbb{F}_2[b, b^{-1}, c]$ and $k_0[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}, b, d, d^{-1}]$, we have $K_1(k_0/(2)) = \{b^m\}$ and $K_1(k_0[\frac{1}{2}]) = \{\pm d^m 2^n\}$. The following diagram shows that $\pi_*(b) = [\begin{smallmatrix}2\\b\end{smallmatrix}]$:

$$\begin{tikzcd}
0 \arrow[r] & k_0^2 \arrow[r, "\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\cong", "\begin{pmatrix} b & 1 \\ 4c & b \end{pmatrix}"'] & k_0^2 \arrow[r] \arrow[d, "\begin{pmatrix} b & 2 \\ 2c & b \end{pmatrix}", "\cong"'] & k_0/(2) \arrow[r] \arrow[d, "b", "\cong"'] & 0 \\
0 \arrow[r] & k_0^2 \arrow[r, "\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}"'] & k_0^2 \arrow[r] & k_0/(2) \arrow[r] & 0.
\end{tikzcd}$$

Since $K_2(\mathbb{Z}[\frac{1}{2}]) = K_2(\mathbb{Z})$, $K_2(k_0[\frac{1}{2}])$ is generated by the symbols $\{-1, 1\}$, $\{-1, d\}$, and $\{2, d\}$. Using the fact that ∂ is the tame symbol for the prime (2) of k_0 (this will be shown in Lemma 1.8 below), we have $\partial\{-1, -1\} = \partial\{-1, d\} = 0$ and $\partial\{2, d\} = d^{-1} \equiv b^{-2} \pmod 2$. This establishes:

Lemma 1.7. *The units of k_0 are $\{\pm d^m\}$, while the unique nonzero element of* $\mathrm{SK}_1(k_0)$ *is $[\begin{smallmatrix}2\\b\end{smallmatrix}] = \pi_*(b)$. The units of l_0 are $\{\pm(b + 2\alpha)^m\}$ and* $\mathrm{SK}_1(l_0) = 0$.

There remains only the claim about ∂. This holds in a more general setting than we need at the moment.

Lemma 1.8. *Let R be a regular domain, (f) a principal prime ideal such that R/f is* *regular. Then the boundary map $\partial: K_2(R[f^{-1}]) \to K_1(R/f)$ in the localization sequence is the tame symbol, in the following sense: if g, h are units in $R[f^{-1}]$, then $\partial\{g,h\} = (-1)^{v(g)v(h)} g^{v(h)}/h^{v(g)}$, where v is the (f)-adic valuation.*

Proof. It is an elementary exercise in commutative algebra that the units of $R[f^{-1}]$ have the form $g = uf^n$, u a unit of R. By additivity, it is enough to show that the formula holds for $\{u,f\}$ and $\{f,f\}$. The element $\partial\{g,h\}$ is described in Proposition 7.7 of [7]. By (7.8), (7.11) of [7] we have $\partial\{u,f\} = [R/f, u] = u$ and $\partial\{f,f\} = [R/f, -1] = -1$. □

Remark. The proof actually proves the following more general proposition. Let R be an integral domain with (f) a principal prime ideal. Then the boundary map $\partial: K_2(R[f^{-1}]) \to K_1(\mathbf{H}_f)$ in the localization sequence of [6] is the tame symbol on Steinberg symbols $\{g,h\}$, composed with the natural map $K_1(R/f) \to K_1(\mathbf{H}_f)$. Regularity of R and R/f allows us to identify $K_1(R/f)$ and $K_1(\mathbf{H}_f)$.

We have applied Lemma 1.8 with $R = k_0$, $f = 2$. In this case $K_2(R[f^{-1}])$ is generated by Steinberg symbols, so ∂ is completely described. We postpone the computation of K_2 and K_3 of k_0, l_0 to Section 2, where we need them.

We now return to the discussion of φ.

Theorem 1.9. *The element $\varphi(1) = \lambda_*^A(u)$ $(1 \in K_0(l))$ lies in $\mathrm{SK}_1(A)$ and equals $[{y \atop x}][{\varepsilon(y) \atop \varepsilon(x)}]^{-1} = \chi \cdot \eta$, where χ is either -1 or $-d$. If $2 = 0$ then $\varphi(1) = 0$. If 2 is a unit of k or if $\mathrm{SK}_1(k) = 0$, then $\varphi(1) = (-1) \cdot \eta = (-d) \cdot \eta$.*

Proof. Relative to the basis $(1, -\alpha)$ of $A \otimes l$ we have

$$u = (x - \alpha y)\varepsilon(x - \alpha y)^{-1} = \left[\begin{pmatrix} x & -cy \\ y & (x + by) \end{pmatrix}\right] \left[\begin{pmatrix} \varepsilon(x) & -c\varepsilon(y) \\ \varepsilon(y) & \varepsilon(x + by) \end{pmatrix}\right]^{-1}.$$

(For the first equality, recall that u is normalized so that $(\varepsilon \otimes l)(u) = 1$.) It follows that the 'norm' of u in $K_1(A)$ is

$$\varphi(1) = \lambda_*^A(u) = \begin{bmatrix} y \\ x \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon(y) \\ \varepsilon(x) \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \begin{bmatrix} \varepsilon(y) \\ \varepsilon(x) \end{bmatrix}^{-1}$$

since $x^2 \equiv d \pmod{y}$. By Proposition 1.5, $\partial\varphi(1) = 1 - 1 = 0$, so for some $\chi \in K_1(k)$ we must have $\varphi(1) = \varepsilon_*(\chi) = \chi \cdot \eta$.

To compute χ, note that (by the discussion in 0.3) there is a map $k_0 \to k$ inducing $A_0 \to A$ and inducing our section ε by base change, where $k_0 = \mathbb{Z}[b, c, (4c - b^2)^{-1}]$. This means that $\chi \in K_1(k_0)$, and is determined up to $\lambda_* K_1(l_0)$.

By Lemma 1.7, $K_1(l_0) = \{\pm(b + 2\alpha)^m\}$ and $K_1(k_0) = \{\pm d^m\} \oplus \mathrm{SK}_1(k_0)$, the only nonzero element of $\mathrm{SK}_1(k_0)$ being $\pi_*(b)$. Now $\lambda_*(b + 2\alpha) = d + \pi_*(b)$, $\lambda_*(d) = d^2$, and $\lambda_*(-1) = +1$. Therefore $K_1(k_0)/\lambda_* K_1(l_0) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, and χ can be chosen from the set of representatives $\{\pm 1, \pm d\}$. On the other hand, the map $k_0 \to \mathbb{R}$

giving the circle scheme (i.e., $b=0$, $c=1$) takes $\varphi(1)$ to $[{}^{y}_{x}]=[{}^{Y}_{X}]$. By [12, p. 129] this is a nontrivial element of $K_1(A\otimes R)$, while $d\cdot\eta$ is mapped to $4\cdot\eta=1\cdot\eta=0$. This means that χ is either -1 or $-d$.

Finally, we note that if $2=0$ we have $d=b^2=\lambda_*(b)$ and $(-1)\cdot\eta=1\cdot\eta=0$ so $\varphi(1)=0$. If $\frac{1}{2}\in k$ then $d=\lambda_*(b+2\alpha)$ so $(-d)\cdot\eta=(-1)\cdot\eta$. If $\mathrm{SK}_1(k)=0$ then $\pi_*(b)$ maps to 0 in $K_1(k)$ so $\lambda_*(b+2\alpha)=d$ and again $(-d)\cdot\eta=(-1)\cdot\eta$. We have not been able to decide if $\chi=-1$ or $\chi=-d$. The observation that if $\mathrm{SK}_1(k)=0$ then $(-d)\cdot\eta=(-1)\cdot\eta$ shows that the question of whether χ is -1 or $-d$ is a subtle one, because we can't test by mapping to a simple known example. □

Theorem 1.10. *The following diagram commutes, with the bottom row (but not the top row) exact:*

$$\begin{tikzcd}
K_n(l) \arrow[r, "\lambda_*"] \arrow[d, "\lambda^*(\chi)"] & K_n(k) \arrow[r, "\lambda_*"] \arrow[d, "\chi"] & K_n(l) \arrow[d, "\varphi"] \arrow[dr, "\zeta-\bar\zeta"] & & \\
K_{n+1}(l) \arrow[r, "\lambda_*"] & K_{n+1}(k) \arrow[r, "\eta"] & \tilde K_{n+1}(A) \arrow[r, "\partial"] & K_n(l) \arrow[r, "\lambda_*"] & K_n(k).
\end{tikzcd}$$

Proof. Exactness of the bottom row is Theorem 1.1. The left loop commutes by the projection formula, and the right loop commutes by Proposition 1.5. Finally, if $\xi\in K_n(k)$, the projection formula yields $\varphi\lambda^*(\xi)=\lambda_*^A\{\lambda^*(\xi),u\}=\xi\cdot\lambda_*^A(u)$, which is $\{\xi,\chi\}\eta$ by Theorem 1.9. □

Remark. One consequence is that $\lambda^*\lambda_*K_n(l)=\{\zeta+\bar\zeta\mid\zeta\in K_nl\}$ is contained in the kernel of φ. We will see in the next section that it is sometimes (but not always) the kernel for $n=1$. When k, l are fields, Theorems 1.9 and 1.11 show that $\varphi: K_0l/\lambda^*\lambda_*K_0(l)\to K_1(A)$ is injective if and only if -1 is not a norm in k^*.

If k is a field but $f(t)$ factors, then $l=k\times k$ and $A=k[z,z^{-1}]$. In this case $\varphi: K_n(l)/\lambda^*\lambda_*K_n(l)=K_n(k)\to\tilde K_{n+1}(A)$ is an isomorphism (as may be seen from Proposition 1.5).

Theorem 1.11. *If k, l are fields, then:*

$$\tilde K_0(A)=\{0,\eta\},\qquad \tilde K_1(A)=\mathrm{SK}_1(A)=k^*/\mathrm{Norm}(l^*)\cdot\eta,$$

and

$$\tilde K_2(A)=\varphi(l^*)\oplus E\cdot\eta,\qquad \mathrm{Norm}(l^*)\subseteq\ker(\varphi)\subseteq k^*,$$

and $E\cong K_2(k)/(\lambda_*K_2l+\{k^*,-1\})$ is an elementary 2-group.

Proof. The formulas for $\tilde K_0$, $\tilde K_1$ follow from Corollary 1.2. By Hilbert's Theorem 90, the map $\partial\varphi: l^*\to\mathrm{Ker}(\mathrm{Norm}: l^*\to k^*)$ is onto. Thus $\tilde K_2(A)$ is generated by $\varphi(l^*)$ and $K_2(k)/\lambda_*K_2(l)\cdot\eta$. Hilbert's Theorem 90 also implies that $\partial\varphi$ has kernel k^*, so by Theorem 1.10 we have $\varphi(l^*)\cap K_2(k)/\lambda_*K_2(l)\cdot\eta=\varphi(k^*)=\{\chi,k^*\}=\{-1,k^*\}$. Since $K_2(k)/\lambda_*K_2(l)\cdot\eta$ is an elementary 2-group, it splits as $\{-1,k^*\}\oplus E$ for some E. Thus $\tilde{K}_2(A)$ is generated by $\varphi(l^*)$ and E, where $\varphi(l^*)\cap E=\{-1,k^*\}\cap E=0$. This gives the desired direct sum decomposition of $\tilde{K}_2(A)$. The inclusion $\mathrm{Norm}(l^*)\subseteq\ker\varphi$ follows from Theorem 1.10, and $\ker\varphi\subseteq\ker\partial\varphi=k^*$. □

Remark. The computation of $\mathrm{SK}_1(A)$ was already given in [15], at least for the circle scheme (Example 0.1(b)). We will give an example in which $E\neq 0$ in Section 2.

Remark 1.12. In general, consider the following conditions:

$(\mathrm{H90}_n)'$: Every $\xi\in K_{n-1}(l)$ with $\lambda_*(\xi)=0$ has the form $\zeta-\bar{\zeta}$ for some $\zeta\in K_{n-1}(l)$.

$(\mathrm{H90}_n)''$: $\lambda^*K_{n-1}(k)=\{\xi\in K_{n-1}(l)\mid\xi=\bar{\xi}\}$.

If $(\mathrm{H90}_n)'$ holds, the proof of Theorem 1.11 gives $\tilde{K}_n(A)=\mathrm{Im}\,\varphi\oplus E\eta$, where E is an elementary 2-group. If $(\mathrm{H90}_n)''$ also holds, then in fact E is isomorphic to $K_n(k)/(\lambda_*K_n(l)+\chi\cdot K_{n-1}(k))$. We will use this remark in Section 2.

2. Examples

In this section we provide examples with $k=\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}(t)$ to show that Theorem 1.11 is the best possible. We then analyse the generic cases, and end by computing the K-theory of the torus.

We first analyze the circle scheme over $\mathbb{R}$, i.e. $A=\mathbb{R}[X,Y]/(X^2+Y^2-1)$. To do this, recall that $K_2\mathbb{C}$ is a $\mathbb{Q}$-vector space, and that the action of complex conjugation has two eigenvalues, ±1. This induces a decomposition: $K_2\mathbb{C}=K_2^+(\mathbb{C})\oplus K_2^-(\mathbb{C})$. Moreover [16], $\lambda^*:K_2\mathbb{R}\to K_2^+(\mathbb{C})$ is onto with kernel $\mathbb{Z}/2$, and is split by $\lambda_*/2$. Thus the $(\mathrm{H90}_3)$-conditions both hold.

The sequence of Corollary 1.2 for $n=2$ becomes

$$0\to\mathbb{Z}/2\xrightarrow{\eta}\tilde{K}_2(A)\xrightarrow{\partial}S^1\to 0.$$

The following result shows that this sequence does not split, answering a question of Roberts [15].

Theorem 2.1. *The map φ induces isomorphisms $\tilde{K}_0(A)=\mathrm{SK}_1(A)=\mathbb{Z}/2$, $K_2(A)=K_2(\mathbb{R})\oplus S^1$, and $K_3(A)=K_3(\mathbb{R})\oplus K_2^-(\mathbb{C})\oplus(K_3(\mathbb{R})/\lambda_*K_3\mathbb{C})\eta$. The element $\varphi(-1)=\{-1,-1\}\eta=(-1)[\tfrac{Y}{X}]$ of $K_2(A)$ is nonzero. (This answers the question raised in* [12, p. 129].)

Proof. We apply Theorem 1.11. For K_2 we have $E=0$, and $\mathrm{Norm}(\mathbb{C}^*)=\mathbb{R}^*_{>0}\subseteq\ker(\varphi)$. By Proposition 1.5, $\partial\varphi:\mathbb{C}^*/\mathbb{R}^*_{>0}=S^1\to\ker(\mathrm{Norm}:\mathbb{C}^*\to\mathbb{R}^*)=S^1$ is multiplication by 2. ($\mathbb{R}^*_{>0}$ is the multiplicative group of positive real numbers.) By Theorem 1.10, $\varphi(-1)=\{-1,-1\}\cdot\eta$, and this is nonzero because it is the generator of the copy of $\mathbb{Z}/2$ in the above sequence from Corollary 1.2. This completes the analysis for K_2.

To compute $\tilde{K}_3(A)$, we note that $\partial\varphi$ is the projection of $K_2(\mathbb{C})$ onto $K_2^-(\mathbb{C})$, followed by multiplication by 2. But $K_2^+(\mathbb{C})=\lambda^*\lambda_*K_2(\mathbb{C})$, so by the remark after Theorem 1.10 we have $\varphi(K_2^+(\mathbb{C}))=0$. Thus $\mathrm{Im}(\varphi)\cong K_2^-(\mathbb{C})$. By Remark 1.12, $\tilde{K}_3(A)=K_2^-(\mathbb{C})\oplus K_3(\mathbb{R})/(\lambda_*K_3\mathbb{C}+\chi\cdot K_2\mathbb{R})$. Since $\varphi(\lambda^*K_2(\mathbb{R}))=\varphi(K_2^+(\mathbb{C}))=0$, we see by Theorem 1.10 that $\chi\cdot K_2(\mathbb{R})$ maps to zero in $\tilde{K}_3(A)$, and therefore that $\chi\cdot K_2(\mathbb{R})\subseteq\lambda_*K_3(\mathbb{C})$. The stated decomposition of $\tilde{K}_3(A)$ follows. □

Remark. If $k=\mathbb{R}$, the construction in Section 0 can produce only two rings: the circle scheme and $\mathbb{R}[t,t^{-1}]$. This follows from the observation after Example 0.1, since $\mathbb{R}$ has only one quadratic extension field.

Remark. We do not know if $\lambda_*:K_3\mathbb{C}\to K_3\mathbb{R}$ is onto. One interesting fact, shown in the proof of Theorem 2.1, is that $\{-1,-1,-1\}=\lambda_*(x)$ for some x (this is nonzero by [9]).

Since we will need it, we insert the following result.

Proposition 2.2. *If $\mathbb{Z}\subseteq k\subseteq\mathbb{Q}$ then $K_3(k)=\mathbb{Z}/48$.*

Proof. By [11], $K_3(\mathbb{Z})=\mathbb{Z}/48$. The fact that $K_3\mathbb{Z}\to K_3(k)$ is onto follows from the localization sequence. The order 24 subgroup is $\mathrm{Im}(J)$, and Giffen [9] has shown that $\mathrm{Im}(J)$ injects into $K_3(\mathbb{R})$, a fortiori into $K_3(k)$. □

The kernel of $\varphi:l^*\to K_2(A)$ can be strictly larger than the norms Nl^*. This is illustrated by the following example, where $N\mathbb{Q}(\mathrm{i})^*\subsetneq\mathbb{Q}^*_{>0}$ (the multiplicative group of positive rational numbers).

Theorem 2.3. *When $k=\mathbb{Q}$, $l=\mathbb{Q}(\mathrm{i})$, we have (for the circle scheme)*

$$\mathrm{SK}_1(A)=\mathbb{Q}^*/N\mathbb{Q}(\mathrm{i})^*=\textit{countable, infinite 2-group},$$

$$K_2(A)=K_2(\mathbb{Q})\oplus(\mathbb{Q}(\mathrm{i})^*/\mathbb{Q}^*_{>0}),$$

$$K_3(A)=K_3(\mathbb{Q})\oplus X\oplus Y$$

where

$$X=K_2\mathbb{Q}(\mathrm{i})/\lambda^*K_2\mathbb{Q}=\coprod_{p\equiv 1(4)}(\mathbb{F}_p^*)\oplus\coprod_{p\equiv 3(4)}(\mathbb{F}_{p^2}^*/\mathbb{F}_p^*)$$

$$Y=K_3\mathbb{Q}/\lambda_*K_3\mathbb{Q}(\mathrm{i})=\textit{either } 0 \textit{ or } \mathbb{Z}/2.$$

Proof. We know SK_1 by Theorem 1.11. In order to compute K_2 and K_3 we need to analyze $K_2\mathbb{Q}$ and $K_2\mathbb{Q}(\mathrm{i})$. Since $K_2\mathbb{Z}[\mathrm{i}]=0$ [22, p. A8], the localization sequences for $\mathbb{Z}$ and $\mathbb{Z}[\mathrm{i}]$ give the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & K_2\mathbb{Q}(\mathrm{i}) \arrow[r] \arrow[d, "\lambda_*"] & \coprod_{\mathfrak{p}} (\mathbb{Z}[\mathrm{i}]/\mathfrak{p})^* \arrow[r] \arrow[d, "\coprod N"] & 0 \\ 0 \arrow[r] & K_2\mathbb{Z} \arrow[r] & K_2\mathbb{Q} \arrow[r] & \coprod_p \mathbb{F}_p^* \arrow[r] & 0 \end{tikzcd} \tag{2.4}$$

We claim that $\coprod N$ is onto. There are two cases to consider. If $p\equiv 1$ (4), $p\mathbb{Z}[\mathrm{i}]=\mathfrak{p}_1\mathfrak{p}_2$ and the norm map is $N=(1,1):\mathbb{F}_p^*\times\mathbb{F}_p^*\to\mathbb{F}_p^*$. If $p\equiv 3$ (4), p does not split and the norm map is $N:\mathbb{F}_{p^2}^*\to\mathbb{F}_p^*$. In either case N is onto, whence the claim. (Note that $\mathbb{F}_2^*=(\mathbb{Z}[\mathrm{i}]/(1+\mathrm{i}))^*=1$ also.)

The next claim is that

$$K_2\mathbb{Q}(\mathrm{i}) \xrightarrow{\lambda_*} K_2\mathbb{Q} \xrightarrow{(\ ,\ )_\infty} \{\pm 1\}\to 0 \tag{2.5}$$

is exact, where $(\ ,\ )_\infty$ is the symbol of [12, p. 104]. That the composition is zero follows from the diagram

$$\begin{tikzcd} K_2\mathbb{Q}(\mathrm{i}) \arrow[r, "\lambda_*"] \arrow[d] & K_2\mathbb{Q} \arrow[d] & \\ K_2\mathbb{C} \arrow[r, "\lambda_*"] & K_2\mathbb{R} \arrow[r, "(\ ,\ )_\infty"] & \{\pm 1\}. \end{tikzcd}$$

The square commutes by Lemma 1.3, and the composition in the bottom row is zero since $K_2\mathbb{C}$ is divisible. (In fact, the bottom row is exact.) Also, $(-1,-1)_\infty=-1$ in (2.5), so we need only show that if $x\in K_2\mathbb{Q}$ and $(\ ,\ )_\infty(x)=1$, then $x=\lambda_*(y)$ for some $y\in K_2\mathbb{Q}(\mathrm{i})$. We saw above that there exists $z\in K_2\mathbb{Z}$ and $y\in K_2\mathbb{Q}(\mathrm{i})$ such that $\lambda_*(y)=x+z$. If we apply $(\ ,\ )_\infty$ – which splits the inclusion $K_2\mathbb{Z}\to K_2\mathbb{Q}$ – we conclude that $z=0$ and $x=\lambda_*(y)$, establishing exactness of (2.5).

As noted in the proof of Theorem 1.11, the kernel of $\partial\varphi$ is $\mathbb{Q}^*$. By (2.5) and Theorem 1.10 $\varphi(\mathbb{Q}^*_{>0})=0$, yet $\varphi(-1)=\{-1,-1\}\eta\neq 0$. Thus the kernel of φ is $\mathbb{Q}^*_{>0}$. Finally $E=0$, so the computation of $\tilde{K}_2(A)$ follows from Theorem 1.11.

The conditions ($\mathrm{H90}_3$) follow by applying Hilbert's Theorem 90 to the components of $\coprod N$. By Remark 1.12, $\tilde{K}_3(A)=\operatorname{Im}\varphi\oplus Y$, where $Y=K_3(\mathbb{Q})/(\lambda_*K_3\mathbb{Q}(\mathrm{i})+\{K_2(\mathbb{Q}),-1\})$. We claim that $\{K_2\mathbb{Q},-1\}\subseteq\lambda_*K_3\mathbb{Q}(\mathrm{i})$. If $x\in\lambda_*K_2\mathbb{Q}(\mathrm{i})$, then $\{x,-1\}\in\lambda_*K_3\mathbb{Q}(\mathrm{i})$ by the projection formula. By (2.5), $\{-1,-1\}$ represents $K_2\mathbb{Q}/\lambda_*K_2\mathbb{Q}(\mathrm{i})$, while $\{-1,-1,-1\}\in\lambda_*K_3\mathbb{Q}(\mathrm{i})$ by diagram chasing in Theorem 1.10 (using the fact that $0=\{-1,-1\}$ in $K_2\mathbb{Q}(\mathrm{i})$). As $K_2(\mathbb{Q})$ is cyclic, Y is cyclic. As $2Y=0$, Y must be either 0 or $\mathbb{Z}/2\mathbb{Z}$.

It remains to show that $\operatorname{Im}\varphi=X$, i.e., that $\lambda^*K_2\mathbb{Q}$ is exactly the kernel of φ. That $\varphi\lambda^*K_2\mathbb{Q}=0$ follows from $\{K_2\mathbb{Q},-1\}\subseteq\operatorname{Im}\lambda_*$ (proved in the preceding paragraph) and Theorem 1.10. On the other hand, $\ker\varphi \subseteq \ker\partial\varphi$, and the latter equals $\lambda^* K_2\mathbb{Q}$ by ($H90_3$). □

Remark. If we replace $l=\mathbb{Q}(\mathrm{i})$ by some other quadratic extension, one would expect similar results. $\tilde{K}_2(A)$ is either $l^*/\mathbb{Q}^*$ or $l^*/\mathbb{Q}^*_{>0}$, depending on whether or not the image of $\lambda_*: K_2(l)\to K_2(\mathbb{Q})$ contains $\{-1,-1\}$. With $\tilde{K}_3(A)$ there could be some difficulty because nontrivial wild symbols in $K_2(l)$ may cause ($H90_3$) to fail.

2.6. We now give an example in which φ does not map onto $\tilde{K}_2(A)$. We take the circle scheme over $k=\mathbb{Q}(t)$, so that $l=\mathbb{Q}(\mathrm{i},t)$. By Theorem 1.11 we have to show that $E=K_2(k)/(\lambda_*K_2 l+\{k^*,-1\})$ is nonzero. Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_2\mathbb{Q}(\mathrm{i}) \arrow[r] \arrow[d, "\lambda_*"] & K_2\mathbb{Q}(\mathrm{i},t) \arrow[r] \arrow[d, "\lambda_*"] & \coprod_P l(P)^* \arrow[r] \arrow[d, "\coprod N"] & 0 \\
0 \arrow[r] & K_2(\mathbb{Q}) \arrow[r] & K_2\mathbb{Q}(t) \arrow[r] & \coprod_{\mathfrak{p}} k(\mathfrak{p})^* \arrow[r] & 0,
\end{tikzcd}$$

where $\mathfrak{p}$ (resp. P) ranges over the nonzero primes of $\mathbb{Q}[t]$ (resp. $\mathbb{Q}(\mathrm{i})[t]$). It follows from (2.5) that $K_2(\mathbb{Q})$ is in the subgroup $\lambda_*K_2 l+\{k^*,-1\}$, so that E is a quotient of $\coprod k(\mathfrak{p})^*$. If $f\in\mathbb{Q}(t)^*$ then the $\mathfrak{p}$-coordinate of $\{-f,-1\}$ is the tame symbol $(-1)^{\nu(f)}$. Since each $\mathfrak{p}$ is principal, it follows that the image of $\{k^*,-1\}$ in $\coprod k(\mathfrak{p})^*$ is $\coprod\{\pm 1\}$. If $\mathfrak{p}$ is split and P lies above $\mathfrak{p}$, then $Nl(P)^*=k(\mathfrak{p})^*$, so it is enough to consider nonsplit $\mathfrak{p}$. Hence

$$E=\coprod_{\text{nonsplit }\mathfrak{p}} k(\mathfrak{p})^*/(\pm Nl(P)^*),$$

which is a countably infinite 2-group. (Even $\mathfrak{p}=(t)$ gives an infinite contribution.)

The generic circle scheme has $k=\mathbb{Z}[\frac{1}{2}]$, $l=k[\mathrm{i}]$. Recall that Corollary 1.2 yields an exact sequence

$$0\to K_n(k)/\lambda_*K_n(l)\xrightarrow{\varepsilon_*}\tilde{K}_n(A)\to\operatorname{Ker}(K_{n-1}(l)\xrightarrow{\lambda_*}K_{n-1}(k))\to 0.$$

We can use this for $0\le n\le 3$ to calculate $\tilde{K}_n(A)$. First of all, $K_0(l)=K_0(k)=\mathbb{Z}$ and λ_* is multiplication by 2. Thus $\tilde{K}_0(A)=\mathbb{Z}/2$. Next, $\mathrm{SK}_1(k)=\mathrm{SK}_1(l)=0$, so $K_1(k)=k^*=\{\pm 2^n\}$ and $K_1(l)=l^*=\{\mathrm{i}^m(1+\mathrm{i})^n\}$. We have $\lambda_*(\mathrm{i})=1$, $\lambda_*(1+\mathrm{i})=2$, so $K_1(k)/\lambda_*K_1(l)=\mathbb{Z}/2$ (generated by -1) and $\tilde{K}_1(A)=\mathrm{SK}_1(A)=\mathbb{Z}/2$. The kernel of $\lambda_*: K_1(l)\to K_1(k)$ is $\mathbb{Z}/4$ (generated by i). The localization sequences yield $K_2(k)=K_2(\mathbb{Z})$ and $K_2(l)=0$, so Corollary 1.2 becomes

$$0\to\mathbb{Z}/2\to\tilde{K}_2(A)\to\mathbb{Z}/4\to 0.$$

To determine the extension, we observe that $\partial\varphi(1+\mathrm{i})=(1+\mathrm{i})/(1-\mathrm{i})=\mathrm{i}$, and $\varphi(1+\mathrm{i})^4=\varphi(-4)=\varphi(-1)=\{-1,-1\}\eta$, the generator of the left hand copy of $\mathbb{Z}/2$.

For $n=3$ the sequence becomes

$$K_3(l) \xrightarrow{\lambda_*} \mathbb{Z}/48 \to \tilde{K}_3(A) \to 0,$$

from which we see that $\tilde{K}_3(A)=Y'$ where $Y'=0$ or $\mathbb{Z}/2$. In fact Y' is the same as the Y of Theorem 2.3. This can be seen by diagram chasing in the following commutative diagram:

$$\begin{tikzcd}
K_3\mathbb{Z}[\frac{1}{2}, \mathrm{i}] \arrow[r, "\lambda_*"] \arrow[d, "\text{onto}"] & K_3\mathbb{Z}[\frac{1}{2}] \arrow[r] \arrow[d, "\cong"] & Y' \arrow[r] \arrow[d] & 0 \\
K_3\mathbb{Q}(\mathrm{i}) \arrow[r, "\lambda_*"] & K_3(\mathbb{Q}) \arrow[r] & Y \arrow[r] & 0.
\end{tikzcd}$$

The left vertical arrow is onto by the corollary to Theorem 5 on p. 113 of [13] (K_2 of a finite field being zero). Altogether, we have:

Proposition 2.7. *Let $A=\mathbb{Z}[\frac{1}{2}][x,y]/(x^2+y^2-1)$. Then $K_0(A)=\mathbb{Z}\oplus\mathbb{Z}/2$, $A^*=\mathbb{Z}[\frac{1}{2}]^*$, $\mathrm{SK}_1(A)=\mathbb{Z}/2$, $K_2(A)=K_2(\mathbb{Z})\oplus\mathbb{Z}/8$ and $K_3(A)=K_3\mathbb{Z}[\frac{1}{2}]\oplus Y$. ($Y$ is the group of 2.3 and is either 0 or $\mathbb{Z}/2$.)*

Now we analyze the generic cases. We have $k_0=\mathbb{Z}[b,c,(4c-b^2)^{-1}]$, $l_0=k_0[\alpha]=\mathbb{Z}[\alpha,b+2\alpha,(b+2\alpha)^{-1}]$, and $A_0=k_0[x,y]/(x^2+bxy+cy^2-d)$ as in Example 0.1(a). Here $c=-\alpha(\alpha+b)$ and $d=4c-b^2=-(b+2\alpha)^2$. The generic case for $2=0$ occurs with $k_0/(2)=\mathbb{F}_2[c,b,b^{-1}]$, $l_0/(2)=\mathbb{F}_2[\alpha,b,b^{-1}]$, and $A_0/(2)$.

Proposition 2.8. *$\tilde{K}_0(A_0/(2))=\mathbb{Z}/2$ on η, $\tilde{K}_1(A_0/(2))=\mathrm{SK}_1(A_0/(2))=\mathbb{Z}/2$ on $b\cdot\eta$, and $\tilde{K}_n(A_0/(2))=0$ for $n\geq 2$.*

Proof. Observe that $K_0(k_0/(2))=K_0(l_0/(2))=\mathbb{Z}$, $K_1(k_0/(2))=K_1(l_0/(2))=\mathbb{Z}$ generated by the unit b. For $n\geq 2$, $K_n(k_0/(2))=K_n(l_0/(2))=K_n(\mathbb{F}_2)\oplus K_{n-1}(\mathbb{F}_2)$, which is a cyclic group of odd order by Theorem 8 of [14]. As $\lambda_*\lambda^*$ is multiplication by 2, λ_* is an injection on all K_n and an isomorphism for $n\geq 2$. For $n=0,1$ the cokernel of λ_* is $\mathbb{Z}/2$. The computation of $\tilde{K}^*(A_0/(2))$ is immediate from Corollary 1.2. $\square$

The generic case when 2 is a unit occurs with $k_0[\frac{1}{2}]=\mathbb{Z}[\frac{1}{2},b,d,d^{-1}]$, $l_0[\frac{1}{2}]=\mathbb{Z}[\frac{1}{2},b,b+2\alpha,(b+2\alpha)^{-1}]$, and $A_0[\frac{1}{2}]$.

Proposition 2.9. *For all $n\geq 0$ there are exact sequences*

$$0\to\mathbb{Z}/2\otimes K_n(\mathbb{Z}[\tfrac{1}{2}]) \xrightarrow{\eta} \tilde{K}_n(A_0[\tfrac{1}{2}]) \xrightarrow{\partial} \mathrm{Tor}(K_{n-1}(\mathbb{Z}[\tfrac{1}{2}]),\mathbb{Z}/2)\to 0.$$

For low n we have

$$\tilde{K}_0(A_0[\tfrac{1}{2}])=\mathbb{Z}/2 \text{ on } \eta,$$

$\tilde{K}_1(A_0[\frac{1}{2}]) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *on* $(-1)\cdot\eta$ *and* $2\cdot\eta$,

$\tilde{K}_2(A_0[\frac{1}{2}]) = \mathbb{Z}/4$ *on generator* $\varphi(b+2\alpha)$,

$\tilde{K}_3(A_0[\frac{1}{2}]) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *on* $(K_3\mathbb{Z})\cdot\eta$ *and* $\varphi\{-1, b+2\alpha\}$,

$\tilde{K}_4(A_0[\frac{1}{2}]) = (\mathbb{Z}/2 \otimes K_4\mathbb{Z}) \oplus \mathbb{Z}/2$ *on* $(K_4\mathbb{Z})\cdot\eta$ *and* $\varphi\{-1,-1,b+2\alpha\}$.

Remark. For $n \geq 2$ we have $\mathbb{Z}/2 \otimes K_n\mathbb{Z} = \mathbb{Z}/2 \otimes K_n(\mathbb{Z}[\frac{1}{2}])$, and for all n we have $\mathrm{Tor}(K_n\mathbb{Z}, \mathbb{Z}/2) = \mathrm{Tor}(K_n\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}/2)$. This follows from the localization sequence of $(\mathbb{Z}, \mathbb{Z}[\frac{1}{2}])$ and the fact that for $m \geq 1$ the group $K_m(\mathbb{F}_2)$ is cyclic of odd order (by Theorem 8 of [14]).

Proof. Observe that $K_n(k_0[\frac{1}{2}]) = K_n(l_0[\frac{1}{2}]) = K_n(\mathbb{Z}[\frac{1}{2}]) \oplus K_{n-1}(\mathbb{Z}[\frac{1}{2}])$. Let ζ be an element in the $K_n(\mathbb{Z}[\frac{1}{2}])$ summand of $K_n(k_0[\frac{1}{2}])$. The corresponding element in $K_n(l_0[\frac{1}{2}])$ is $\lambda^*\zeta$, and $\lambda_*(\lambda^*\zeta) = 2\zeta$. Using the decomposition in [6], the elements in $K_{n+1}(k_0[\frac{1}{2}])$ and $K_{n+1}(l_0[\frac{1}{2}])$ corresponding to ζ are $\{\zeta, d\}$ and $\{\lambda^*\zeta, b+2\alpha\}$. Note that as $K_1(k_0[\frac{1}{2}])$ consists of units we have $\lambda_*(b+2\alpha) = N(b+2\alpha) = d$. The projection formula yields $\lambda_*\{\lambda^*\zeta, b+2\alpha\} = \{\zeta, \lambda_*(b+2\alpha)\} = \{\zeta, d\}$. We have shown that λ_* acts component-wise as multiplication by 2 and 1, so the short exact sequence of the Proposition follows immediately from Corollary 1.2.

Now $K_0(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}$, $K_1(\mathbb{Z}[\frac{1}{2}]) = \{\pm 2^m\}$, $K_2(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}/2$ on $\{-1,-1\}$, and $K_3(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}/48$ (by Proposition 2.2). This immediately yields $\tilde{K}_0$ and $\tilde{K}_1$ for $A_0[\frac{1}{2}]$, and gives $\tilde{K}_n$ $(n=2,3,4)$ as extensions of $\mathbb{Z}/2 = \mathrm{Tor}(K_{n-1}\mathbb{Z}[\frac{1}{2}], \mathbb{Z}/2)$ on generators -1, $\{-1,-1\}$, and $\{-1,-1,-1\}$ respectively. We lift these elements to $\tilde{K}_n(A_0[\frac{1}{2}])$ by using Proposition 1.5 to observe that $\partial\varphi(b+2\alpha) = (b+2\alpha)/(b+2\bar{\alpha}) = -1$, and similarly $\partial\varphi\{-1, b+2\alpha\} = \{-1,-1\}$, $\partial\varphi\{-1,-1,b+2\alpha\} = \{-1,-1,-1\}$. Now $(b+2\alpha)^2 = -d = (-1)\cdot\lambda_*(b+2\alpha)$, so by Theorem 1.10 we have $2\varphi(b+2\alpha) = \varphi(-d) = \varphi(-1) = \{-1,-1\}\cdot\eta$, which is nonzero. This shows that $\tilde{K}_2(A_0[\frac{1}{2}]) = \mathbb{Z}/4$ on generator $\varphi(b+2\alpha)$. On the other hand, $2\varphi\{-1, b+2\alpha\} = \varphi\{+1, b+2\alpha\} = 0$, and $2\varphi\{-1,-1,b+2\alpha\} = 0$ similarly. This shows that the extensions $\tilde{K}_3$ and $\tilde{K}_4$ of $\mathbb{Z}/2$ by $\mathbb{Z}/2 \otimes K_n(\mathbb{Z}[\frac{1}{2}])$ split. The form given in the Proposition follows from the remark preceding the proof. $\square$

Now we tackle A_0. By the discussion preceding Lemma 1.7 and by Corollary 1.2, $\tilde{K}_0(A_0) = \mathbb{Z}/2$ on η, and $\tilde{K}_1(A_0) = K_1(k_0)/\lambda_*K_1(l_0)$. By Lemma 1.7, $K_1(l_0) = \{\pm(b+2\alpha)^m\}$ and $K_1(k_0) = \{\pm d^m\} \oplus \mathrm{SK}_1(k_0)$ (the latter summand generated by $[\frac{2}{b}]$). As $\lambda_*(-1) = 1$ and $\lambda_*(b+2\alpha) = d[\frac{2}{b}]$, we have $\tilde{K}_1(A_0) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ on $(-1)\cdot\eta$ and $d\cdot\eta$.

Since A_0 is free (hence flat) over k_0, there is a morphism '$\otimes_{k_0}A_0$' between the localization sequences for $(k_0, k_0[\frac{1}{2}])$ and $(A_0, A_0[\frac{1}{2}])$. This yields the commutative diagram (with exact rows and columns):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
K_n(k_0/2) \arrow[r] \arrow[d, "f_n"] & K_n(k_0) \arrow[r] \arrow[d] & K_n(k_0[\frac{1}{2}]) \arrow[r] \arrow[d] & K_{n-1}(k_0/2) \arrow[r] \arrow[d, "f_{n-1}"] & K_{n-1}(k_0) \arrow[d] \\
K_n(A_0/2) \arrow[r] & K_n(A_0) \arrow[r] \arrow[d] & K_n(A_0[\frac{1}{2}]) \arrow[r] \arrow[d] & K_{n-1}(A_0/2) \arrow[r] & K_{n-1}(A_0) \\
& \tilde{K}_n(A_0) \arrow[r] \arrow[d] & \tilde{K}_n(A_0[\frac{1}{2}]). \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd}$$

Proposition 2.8 implies that f_n is an isomorphism for $n \geq 2$. Diagram chasing then shows that $\tilde{K}_n(A_0) \to \tilde{K}_n(A_0[\frac{1}{2}])$ is an isomorphism for $n \geq 3$, and an inclusion for $n=2$. As the generator $\varphi(b+2\alpha)$ of $\tilde{K}_2(A_0[\frac{1}{2}])$ is in the image of $\tilde{K}_2(A_0)$ (by naturality of φ), we see that $\tilde{K}_2(A_0) = \tilde{K}_2(A_0[\frac{1}{2}]) = \mathbb{Z}/4$ as well. Using the remark after Proposition 2.9, we have:

Proposition 2.10. *For $n \geq 2$ there is an exact sequence*

$$0 \to \mathbb{Z}/2 \otimes K_n(\mathbb{Z}) \xrightarrow{\eta} \tilde{K}_n(A_0) \xrightarrow{\partial} \operatorname{Tor}(K_{n-1}(\mathbb{Z}), \mathbb{Z}/2) \to 0.$$

For low n we have

$\tilde{K}_0(A_0) = \mathbb{Z}/2$ *on* η,

$\tilde{K}_1(A_0) = \mathrm{SK}_1(A_0) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *on* $(-1) \cdot \eta$ *and* $d \cdot \eta$,

$\tilde{K}_2(A_0) = \mathbb{Z}/4$ *on generator* $\varphi(b+2\alpha)$,

$\tilde{K}_3(A_0) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *on* $(K_3\mathbb{Z}) \cdot \eta$ *and* $\varphi\{-1, b+2\alpha\}$,

$\tilde{K}_4(A_0) = (\mathbb{Z}/2 \otimes K_4\mathbb{Z}) \oplus \mathbb{Z}/2$ *on* $(K_4\mathbb{Z}) \cdot \eta$ *and* $\varphi\{-1, -1, b+2\alpha\}$.

For $n \geq 2$ the inclusion $A_0 \to A_0[\frac{1}{2}]$ induces an isomorphism on $\tilde{K}_n$.

Preparatory to Section 3, we now compute the K-theory of the torus. For any covariant functor F from k-algebras to abelian groups, define a new functor AF by $AF(B) = \ker(F(B \otimes \varepsilon) : F(B \otimes A) \to F(B))$. Since $B \otimes \varepsilon$ is split by the inclusion of B in $B \otimes A$, we have $F(B \otimes A) = F(B) \oplus AF(B)$. In this notation, $AK_n(k)$ denotes $\tilde{K}_n(A)$.

If we copy the formalism of [3, p. 663], reading AF for NF, we obtain the natural decomposition $F(B\otimes A\otimes A)=(1+A)^2F(B)=F(B)\oplus 2AF(B)\oplus A^2F(B)$. Similarly, $F(B\otimes \bigotimes_{i=1}^m A)=(1+A)^mF(B)=\bigoplus_{i=0}^m \binom{m}{i}A^iF(B)$ for any k-algebra B, and for $B=k$ we have $F(\bigotimes_{i=1}^m A)=\bigoplus_{i=0}^m\binom{m}{i}A^iF(k)$.

Write $C(k)$ for the exact sequence in Theorem 1.1. As this sequence is natural in k, we get a split exact sequence $0\to C(k)\to C(A)\to AC(k)\to 0$, where $AC(k)$ is the sequence

$$\cdots AK_n(l)\xrightarrow{A\lambda_*} AK_n(k)\xrightarrow{A\varepsilon_*} A^2K_n(k)\xrightarrow{\partial} AK_{n-1}(l)\cdots.$$

Because of the splitting, the sequence $AC(k)$ is exact; diagram chasing shows that AC is natural in k. Repeating the process yields natural exact sequences $A^iC(k)$:

$$\cdots A^iK_n(l)\xrightarrow{A^i\lambda_*} A^iK_n(k)\xrightarrow{A^i\varepsilon_*} A^{i+1}K_n(k)\xrightarrow{\partial} A^iK_{n-1}(l)\cdots, \qquad (2.11)$$

which end in $A^iK_0(l)\to A^iK_0(k)\to A^{i+1}K_0(k)\to 0$.

The analysis of the groups A^iK_0 is particularly simple, since we have $K_0(l\otimes\bigotimes_{i=1}^m A)=K_0(l[u_1,u_1^{-1},\dots,u_m,u_m^{-1}])=K_0(l)$. It follows that $A^iK_0(l)=0$ for $i>0$. From the ending of (2.11), we deduce that $A^iK_0(k)=AK_0(k)$ for all $i>0$. In particular, $K_0(A\otimes A)=K_0(k)\oplus 3AK_0(k)$ as a $K_0(k)$-module.

Proposition 2.12. *As a ring, $K_0(A\otimes A)=K_0(A)\otimes_{K_0k}K_0(A)$. As a $K_0(k)$-module, it is generated by 1, $\eta\otimes 1$, $1\otimes\eta$, and $(\eta\otimes 1)(1\otimes\eta)$. More generally, if $\otimes$ runs over any fixed indexing set, then $K_0(\bigotimes_k A)=\bigotimes_{K_0k}K_0(A)$.*

Proof. Let B be any k-algebra. Then $B\otimes_k A$ is an augmented B-algebra with augmentation $B\otimes\varepsilon: B\otimes A\to B$. The structure of $K_0(B\otimes_k A)$ as a $K_0(B)$-algebra is given by Corollary 1.2 and the subsequent remark. That is, $K_0(B\otimes_k A)=K_0(B)\oplus [K_0(B)/(B\otimes\lambda)_*K_0(B\otimes l)]\cdot\mu$, where $\mu^2=0$ and

$$\mu=[B\otimes_k A]-[\ker(B\otimes\varepsilon)]=[B\otimes_k A]-[B\otimes\ker(\varepsilon)]$$

in $K_0(B\otimes A)$. (Recall that $\ker(\varepsilon)$ is a projective A-module.) Thus μ is the image of $\eta=[A]-[\ker(\varepsilon)]$ under the homomorphism $K_0(A)\to K_0(B\otimes A)$ induced by the inclusion $A=k\otimes_k A\to B\otimes_k A$. For this reason we will write $\mu=1\otimes\eta$.

Now suppose that $B=\bigotimes_{i=1}^n A$. We have seen that as a $K_0(k)$-module $K_0(B)=K_0(k)\oplus\bigoplus_{i=1}^n\binom{n}{i}AK_0(k)$. Since $K_0(B\otimes l)=K_0(l)$, we have

$$K_0(B)/(B\otimes\lambda)_*K_0(B\otimes l)=[K_0(k)/\lambda_*K_0(l)]\oplus\bigoplus_{i=1}^n\binom{n}{i}AK_0(k).$$

But $AK_0(k)=K_0(k)/\lambda_*K_0(l)$, so $AK_0(k)\otimes_{K_0k}AK_0(k)=AK_0(k)$ and $K_0(B)/(B\otimes\lambda)_*K_0(B\otimes l)=K_0(B)\otimes_{K_0k}AK_0(k)$ as $K_0(B)$-modules. Thus $K_0(B\otimes A)=K_0(B)\otimes_{K_0k}K_0(A)$ as $K_0(k)$-algebras. The formula for $K_0(\bigotimes A)$ now follows by induction and (for infinite $\otimes$) the fact that K_0 commutes with direct colimits.

In particular, if $B=A$ then $K_0(A\otimes A)=K_0(A)\oplus[K_0(A)/(A\otimes\lambda)_*K_0(A\otimes l)]\cdot(1\otimes\eta)$. By (1.2), $K_0(A)=K_0(k)\oplus[K_0(k)/\lambda_*K_0(l)]\cdot\eta$. But $A\otimes A$ is regarded as an A-algebra via $a\mapsto a\otimes 1$, so this η is identified with $\eta\otimes 1\in K_0(A\otimes A)$. Finally,

$K_0(A\otimes l)=K_0(l)$, so $K_0(A\otimes A)=K_0(k)\oplus AK_0(k)\cdot(\eta\otimes 1)\oplus AK_0(k)\cdot(1\otimes\eta)\oplus AK_0(k)\cdot(\eta\otimes 1)(1\otimes\eta)$. The summands of $K_0(\otimes_{i=1}^n A)$ can be identified in a similar manner. $\square$

Note that we do not claim that $K_0(B\otimes A)=K_0(B)\otimes_{K_0k}K_0(A)$ for any k-algebra B. In fact, this is false for $B=l$.

In order to analyze the higher K-theory of $A\otimes A$, we need a description of the $\binom{m}{i}$ summands $A^iK_n(l)$ of $K_n(l\otimes\otimes_{i=1}^m A)=K_n(l[u_1,u_1^{-1},\dots,u_m,u_m^{-1}])$. Since $l\otimes A=l[u,u^{-1}]$, we have $AK_n(l)=K_{n-1}(l)$, and by induction $A^iK_n(l)=K_{n-i}(l)$. The summand corresponding to the subset $\{j_1,\dots,j_i\}$ of $\{1,\dots,m\}$ is obtained by multiplying $K_{n-i}(l)$ by the element $\{u_{j_1},\dots,u_{j_i}\}$ of $K_i(l[u_1,u_1^{-1},\dots,u_m,u_m^{-1}])$. If we think of the sequence $A^iC(k)$ in (2.11) as a summand of $C(\otimes_{j=1}^i A)$, then $A^iK_n(l)$ is the summand $K_{n-i}(l)\cdot\{u_1,\dots,u_i\}$ of $K_n(l\otimes\otimes A)$. Thus the composite $K_{n-i}(l)=A^iK_n(l)\to A^iK_n(k)\to K_n(\otimes A)$ is the map sending $\zeta\in K_{n-i}(l)$ to $\lambda_*\{\zeta,u_1,\dots,u_i\}$. In particular, the transfer $A\lambda_*$ is just the map φ of Section 1, and we have the long exact sequence

$$\cdots K_{n-1}(l)\xrightarrow{\varphi}AK_n(k)\xrightarrow{(1\otimes\eta)}A^2K_n(k)\xrightarrow{\partial}K_{n-2}(l)\cdots. \tag{2.13}$$

ending in

$$K_0(l)\to AK_1(k)\xrightarrow{(1\otimes\eta)}A^2K_1(k)\to 0.$$

We can use the description of Theorem 1.1 to obtain the following result:

Proposition 2.14. *Assume l is a field, and let η_1, η_2 denote $\eta\otimes 1$, $1\otimes\eta$ in $K_0(A\otimes A)$. Then $(A\otimes A)^*=k^*$,*

$$\mathrm{SK}_1(A\otimes A)=(k^*/Nl^*)\cdot\eta_1\oplus(k^*/Nl^*)\cdot\eta_2\oplus(k^*/\pm Nl^*)\cdot\eta_1\eta_2,$$

$$K_2(A\otimes A)=K_2(k)\oplus(\operatorname{Im}\varphi\times\operatorname{Im}\varphi)\oplus\mathbb{Z}\oplus E\eta_1\oplus E\eta_2\oplus E\eta_1\eta_2.$$

Proof. By Theorem 1.11 and the decomposition $K_n(A\otimes A)=(1+A)^2K_n(k)$, we only have to show that $A^2K_1(k)=(k^*/\pm Nl^*)\cdot\eta_1\eta_2$ and that $A^2K_2(A\otimes A)=\mathbb{Z}\oplus E\eta_1\eta_2$. Notice that the two copies of $AK_n(k)\subseteq K_n(A)$ are embedded via the maps $(A\otimes\iota)^*$, $(\iota\otimes A)^*:K_n(A)\to K_n(A\otimes A)$, where $\iota:k\to A$ is the inclusion. Thus by naturality the elements $\zeta\cdot\eta\in AK_1(k)$ are embedded as $(A\otimes\iota)^*\zeta=\zeta\cdot(A\otimes\iota)^*(\eta)=\zeta\cdot\eta_1$ and as $\zeta\cdot\eta_2$, and similarly for $AK_2(k)$. The ending of (2.13) is

$$\mathbb{Z}\xrightarrow{\varphi}(k^*/Nl^*)\cdot\eta_1\xrightarrow{\eta_2}A^2K_1(k)\to 0,$$

and $\varphi(1)=(-1)\cdot\eta_1$ by Theorem 1.9, so the computation of $A^2K_1(k)$ follows. From (2.13) we get an exact sequence

$$K_1(l)\xrightarrow{\varphi}AK_2(k)\xrightarrow{\eta_2}A^2K_2(k)\xrightarrow{\partial}\mathbb{Z}\xrightarrow{\varphi}(k^*/Nl^*)\cdot\eta_1.$$

But $AK_2(k)=\operatorname{Im}(\varphi)\oplus E\eta_1$, and $\operatorname{Im}(\partial)$ is either $\mathbb{Z}$ or $2\mathbb{Z}$ (depending of whether or not $-1\in\operatorname{Norm}(l^*)$), so $A^2K_2(k)$ must be $E\eta_1\eta_2\oplus\mathbb{Z}$. $\square$

Remark. If -1 is not a norm, then the $\mathbb{Z}$ in $A^2K_2(k)$ is generated by $\lambda_*\{u\otimes 1, 1\otimes u\}$. This follows from Proposition 1.5, since we have $\partial\varphi(u\otimes 1)=(u\otimes 1)/(\bar{u}\otimes 1)=u^2\otimes 1$. If -1 is a norm of an element in l^*, $\lambda_*\{u\otimes 1, 1\otimes u\}$ is not a generator.

Corollary 2.15. *Let $B=\mathbb{R}[x_1,x_2,y_1,y_2]/(x_1^2+y_1^2-1, x_2^2+y_2^2-1)$ be the coordinate ring of the torus. Then:*

$$K_0B=\mathbb{Z}\oplus(\mathbb{Z}/2)^3,$$

$$K_1B=\mathbb{R}^*\oplus(\mathbb{Z}/2)^2,$$

$$K_2B=K_2(\mathbb{R})\oplus(S^1\times S^1)\oplus\mathbb{Z}.$$

K_0B is generated by 1, η_1, η_2 and $\eta_1\eta_2$, $\mathrm{SK}_1(B)$ is generated by

$$(-1)\cdot\eta_i=\begin{bmatrix} y_i \\ x_i \end{bmatrix},\quad i=1,2.$$

The subgroup $(S^1\times S^1)\oplus\mathbb{Z}$ of $K_2(B)$ is obtained by transfer from $B\otimes\mathbb{C}=\mathbb{C}[t_1,t_1^{-1},t_2,t_2^{-1}]$.

3. Primitive elements

In this section we give a partial answer to a question raised by Bloch [1]. In order to clarify matters, we need a short digression.

Let $\mathscr{C}$ denote the category of functors from commutative k-algebras to abelian groups. Bloch [1] has defined an internal hom-functor $\mathrm{Hom}(\ ,\ ):\mathscr{C}\times\mathscr{C}\to\mathscr{C}$. Now $G\in\mathscr{C}$ is called representable if $G(R)=\mathrm{Hom}_{k\text{-alg}}(A,R)$ for some fixed commutative (and therefore cocommutative Hopf) algebra A over k. It turns out that $\mathrm{Hom}(F,G)(k)$ is the subgroup of *primitive* elements of the group $F(A)$, i.e., those elements $x\in F(A)$ for which $\Delta^*(x)=x\otimes 1+1\otimes x$ in $F(A\otimes_k A)$, where Δ is the coproduct and $x\otimes 1$ means $F(A\otimes\iota)(x)$, $1\otimes x$ means $F(\iota\otimes A)(x)$. Bloch has asked for a description of $\mathrm{Hom}(K_*,G)$, i.e., for the primitive elements in $K_*(A)$.

Now the unit $\iota:k\to A$ is split by the counit ε, so if $AF(k)=\ker(\varepsilon^*)$, $F(A)=F(k)\oplus AF(k)$. Moreover, $\varepsilon=(\varepsilon\otimes\varepsilon)\Delta=(\varepsilon\otimes\varepsilon)(A\otimes\iota)=(\varepsilon\otimes\varepsilon)(\iota\otimes A)$, so $(\varepsilon\otimes\varepsilon)^*(x\otimes 1+1\otimes x-\Delta^*(x))=\varepsilon^*(x)$ for any $x\in F(A)$. Thus the primitive elements lie in $AF(k)$.

We now consider the problem of determining the primitives in $K_*(A)$, where A is the ring under study in this paper.

Proposition 3.1. *$\varphi(x)$ is primitive for every $x\in K_*(l)$.*

Proof. $\Delta^*\varphi(x)=\Delta^*\lambda_*^A\{x,u\}=(\lambda^A\otimes\lambda^A)_*\Delta_l^*\{x,u\}$, where $\Delta_l: A\otimes l\to A\otimes A\otimes l$ is the induced coproduct. By 0.4, $\Delta_l^*(u)=\Delta_l(u)=u\otimes u=(u\otimes 1)(1\otimes u)$, while $\Delta_l^*(x)=x$ in $K_*(A\otimes A\otimes l)$. We then compute:

$$\begin{aligned}\Delta^*\varphi(x) &= (\lambda^A \otimes \lambda^A)_*(\{x, u\otimes 1\} + \{x, 1\otimes u\})\\ &= (\lambda^A \otimes \lambda^A)_*(\{x, u\}\otimes 1 + 1\otimes \{x, u\})\\ &= (\lambda^A_*\{x, u\})\otimes 1 + 1\otimes(\lambda^A_*\{x, u\})\\ &= \varphi(x)\otimes 1 + 1\otimes\varphi(x). \qquad \square\end{aligned}$$

One immediate consequence is this: in the circle schemes over $\mathbb{R}$ and $\mathbb{Q}$, every element of $\tilde{K}_2(A)$ is primitive.

We now show that $\mathrm{Im}(\varphi)$ often accounts for all the primitive elements of $K_n(A)$.

Proposition 3.2. *$K_0(A)$ contains no primitive elements (except 0), i.e.* $\mathrm{Hom}(K_0, G) = 0$.

Proof. We first consider the generic case $\eta \in K_0A_0$ of Example 0.1(a). We write $\mathfrak{q} = \ker(\varepsilon)$ so $\eta = 1 - [\mathfrak{q}]$. By Propositions 2.10 and 2.12 we have $\tilde{K}_0(A_0\otimes A_0) = (\mathbb{Z}/2)^3$ on generators $\eta_1 = 1 - [\mathfrak{q}\otimes A_0]$, $\eta_2 = 1 - [A_0\otimes\mathfrak{q}]$, and $\eta_1 + \eta_2 - \eta_1\eta_2 = 1 - [\mathfrak{q}\otimes A_0][A_0\otimes\mathfrak{q}] = 1 - [\mathfrak{q}\otimes\mathfrak{q}]$. It follows that $\det(\eta_1\eta_2) = [\mathfrak{q}\otimes\mathfrak{q}][\mathfrak{q}\otimes A_0]^{-1}[A_0\otimes\mathfrak{q}]^{-1} = 1$. Hence $\mathrm{Pic}(A_0\otimes A_0)$ consists of the four elements 1, $[\mathfrak{q}\otimes A_0]$, $[A_0\otimes\mathfrak{q}]$, and $[\mathfrak{q}\otimes\mathfrak{q}]$.

Now $\Delta^*\eta = 1 - \Delta^*[\mathfrak{q}]$, where $\Delta^*[\mathfrak{q}] \in \mathrm{Pic}(A_0\otimes A_0)$. By the counitary axiom for coalgebras, we must have $[\mathfrak{q}] = (1\otimes\varepsilon)^*\Delta^*[\mathfrak{q}] = (\varepsilon\otimes 1)^*\Delta^*[\mathfrak{q}]$. Inspection of the possibilities shows we must have $\Delta^*[\mathfrak{q}] = [\mathfrak{q}\otimes\mathfrak{q}]$. Thus η is not primitive, as $\eta\otimes 1 + 1\otimes\eta - \Delta^*\eta = \eta_1\eta_2 \neq 0$.

In the general case, the typical element of $\tilde{K}_0(A) = \ker(\varepsilon^*)$ is $x\cdot\eta$ for $x \in K_0k/\lambda_*K_0l$. By naturality we have

$$\begin{aligned}(x\cdot\eta)\otimes 1 + 1\otimes(x\cdot\eta) - \Delta^*(x\cdot\eta) &= x\cdot(\eta\otimes 1) + x\cdot(1\otimes\eta) - x\cdot\Delta^*(\eta)\\ &= x\cdot(\eta_1\eta_2).\end{aligned}$$

By Proposition 2.12, $x\cdot(\eta_1\eta_2) = 0$ iff $x = 0$, and this shows that $x\cdot\eta$ is primitive only when $x = 0$. $\square$

Corollary 3.3. *The primitive elements in $(K_nk)\eta \subseteq K_nA$ are exactly the elements* $\mathrm{Im}(\varphi)\cap(K_nk)\eta$.

Proof. For $x \in K_n(k)$, the same argument shows that $x\cdot\eta$ is primitive iff $x\cdot(\eta_1\eta_2) = 0$ in $A^2K_n(k)$. The exact sequence (2.13) shows that this is so just in case $x\cdot\eta \in \varphi(K_{n-1}l)$. $\square$

In particular, if l is a field then $\varphi(1) = [{}^y_x]$ is the only ($\neq 0$) primitive element of $K_1(A)$, and $\varphi(l^*)$ are the only primitives of $K_2(A)$. More generally, we have (by Remark 1.12):

Corollary 3.4. *The set of primitive elements of* $K_n(A)$ *is exactly* $\mathrm{Im}(\varphi)$ *whenever the condition* $(\mathrm{H90}_n)'$ *of Remark* 1.12 *is satisfied.*

We conclude with the following special case. If $l = k \times k$, so that $A = k[t, t^{-1}]$, then $\tilde{K}_n(A) = K_{n-1}(k) = \mathrm{Im}(\varphi)$. The conditions $(\mathrm{H90}_n)$ are satisfied for each n, and we recover the result of Bloch [1]: the subgroup of primitive elements of $K_n(A)$ is $t \cdot K_{n-1}(k)$.

4. Basepoint independence

In this section we explain the basepoint problem, and solve it (partially) for the circle scheme. We then indicate how this affects the K-theory of schemes obtained by glueing k-rational points together.

The general basepoint problem for schemes X over $S = \mathrm{Spec}(k)$ may be stated as follows. A (k-) rational point of X is a section $p: S \to X$ of the structural morphism. Locally, $X = \mathrm{Spec}(A)$ for a k-algebra A and p is given by an epimorphism $A \to k$, the composite $k \to A \to k$ being the identity. A rational point induces a direct sum decomposition $K_*(X) = K_*(S) \oplus \ker(p^*)$.

Basepoint problem. Determine how the summand $\ker(p^*)$ varies as a subgroup of $K_*(X)$ with the k-rational point p.

As an example, set $X = \mathbb{A}_k^n = \mathrm{Spec}(k[x_1, \ldots, x_n])$. If k is regular, then by [13] $K_*(X) = K_*(k)$, so $\ker(p^*) = 0$ for every p-rational point.

Proposition 4.1. *Fix* $n \geq 1$, $X = \mathbb{A}_k^n$. *The subgroup* $\ker(p^*)$ *of* $K_i(X)$ *is independent of p iff the two maps* '$t=0$', '$t=1$' $: K_i(K[t]) \to K_i(k)$ *agree.*

Proof. *Only if* is clear. Conversely, given p, q and $u \in \ker(p^*)$, we have to show $q^*(u) = 0$. Changing coordinates, we can assume that $p, q: k[x_1, \ldots, x_n] \to k$ are given by $p(x_i) = 0$, $q(x_i) = \alpha_i$. Map $f: k[x_1, \ldots, x_n] \to k[t]$ by $f(x_i) = \alpha_i t$. Then $(t=0)^* f^*(u) = p^*(u) = 0$, so we must have $(t=1)^* f^*(u) = q^*(u)$ zero as well. □

Remark 4.2. The second condition in Proposition 4.1 is equivalent to $K_i(k) = [K_i]k$, where $[K_i]$ denotes the homotopization. See [25]. For $i = 1$, $[K_1]k$ agrees with the Karoubi–Villamayor functor, $\mathrm{KV}_1(k)$. We see from the Gersten–Anderson spectral sequence and a result of Vorst [23] that if k is K_{i-1}-regular ($i \geq 2$) then $[K_i]k = KV_i(k)$. As the Karoubi–Villamayor groups are often easier to compute for non-regular rings, the basepoint problem is related to the problem of computing $K_i(k)$.

Another example of a scheme satisfying basepoint independence is $X = \mathbb{P}_k^1$, k regular. In this case $K_*(X) = K_*(k) \otimes \mathbb{Z}[\mathfrak{o}_p]/(\mathfrak{o}_p^2)$ by a result of Quillen [13, p. 143]. If

p is a k-rational point (a section $p:\mathrm{Spec}(k)\to X$ of the structural morphism), then $p^*(o_p)=0$ [15, Section 3], so $\ker(p^*)=K_*(k)\cdot[o_p]$ is independent of the choice of p.

A simple example in which $\ker(p^*)$ varies is supplied by $X=\mathrm{Spec}(k[t,t^{-1}])$. For simplicity, let us assume that k is regular, so that $K_n(X)\cong K_n(k)\oplus K_{n-1}(k)$. According to [10], a typical element can be written $u+\{v,t\}$, where $u\in K_n(k)$, $v\in K_{n-1}(k)$. Every rational point p is given by sending t to a unit α of k. This sends $u+\{v,t\}$ to $u+\{v,\alpha\}$. Hence:

Proposition 4.3. *If p is the rational point $t=\alpha$ of $X=\mathrm{Spec}\,k[t,t^{-1}]$, multiplication by t/α gives the isomorphism $K_{n-1}(k)\to\ker(p^*)\subseteq K_n(X)$. For $n=1$, $K_0(k)=\mathbb{Z}$, the isomorphism $\mathbb{Z}\cong\ker(p^*)$ is the exponential map $i\to(t/\alpha)^i$. It follows that $\ker(p^*)\subseteq K_1(X)$ is different for every rational point p.*

We turn to the basepoint problem for the circle $X=\mathrm{Spec}(A)$, $A=k[x,y]/(x^2+y^2-1)$, k a field of characteristic $\neq 2$. As $\eta=j^*(o_p)$, the analysis of $\mathbb{P}^1_k$ shows that $K_n(k)\cdot\eta\subseteq\ker(p^*)$, independent of p. Hence the sequence of Corollary 1.2 becomes

$$0\to K_n(k)/\lambda_*K_n(l)\xrightarrow{\ \eta\ }\ker(p^*)\to\ker(\lambda_*)\to 0.$$

As $\ker(\lambda_*)=0$ for $n=0,1$ it follows that X satisfies basepoint independence for K_0, K_1. For K_2, we now assume $k=\mathbb{R}$.

Proposition 4.4. *If $e^{i\theta}$ is a root of unity, the k-rational point $p_\theta=(x-\cos\theta,y-\sin\theta)$ satisfies $p_\theta^*\varphi=0$, and $\varphi:S^1\to\ker(p_\theta^*)\subseteq K_2(A)$ is an isomorphism. Hence $\ker(p_\theta^*)=\ker(p_\zeta^*)$ if $e^{i(\theta-\zeta)}$ is a root of unity.*

Proof. Let $z=e^{i\varphi}\in S^1$. By Lemma 1.3 we have $p_\theta^*\varphi(z)=p_\theta^*\lambda_*^A\{z,u\}=\lambda_*(p_\theta\otimes\lambda)^*\{z,u\}=\lambda_*\{z,e^{i\theta}\}$, and this is zero since $\{z,e^{i\theta}\}=\{z^{1/n},e^{in\theta}\}=\{z^{1/n},1\}=0$. □

We now analyze the case of a general point p_θ. As in the proof of Proposition 4.4 we have $p_\theta^*\varphi(z)=\lambda_*\{z,e^{i\theta}\}$. By [16] we have for $z\in S^1$ that

$$\begin{aligned}\lambda^*p_\theta^*\varphi(z)&=\lambda^*\lambda_*\{z,e^{i\theta}\}\\&=\{z,e^{i\theta}\}\{\bar{z},e^{-i\theta}\}=2\{z,e^{i\theta}\}.\end{aligned}$$

As λ^* is injective on $\lambda_*K_2\mathbb{C}$, and $K_2\mathbb{C}$ is divisible, we conclude that $p_\theta^*\varphi(z)=0$ iff $\{z,e^{i\theta}\}=0$. In particular, if z and $e^{i\theta}$ are algebraically independent over $\mathbb{Q}$, $p_\theta^*\varphi(z)\neq 0$ by a result of Springer [19]. Thus $\ker(p_\theta^*)\neq\ker(p_\phi^*)$ if $e^{i(\theta-\phi)}$ is transcendental.

We now show how the solution of the basepoint problem is useful in computing the (lower) K-groups of 'glued' schemes [17]. For simplicity, we assume that k is a field and A is a regular affine k-algebra. We set $S=\mathrm{Spec}(k)$, $X=\mathrm{Spec}(A)$.

Definition 4.5. Let $p_1, \dots, p_l$ be k-rational points of X. The scheme 'obtained from X by glueing the p_i' is $Y = \operatorname{colim}(\mathbf{Y})$, where $\mathbf{Y}$ is the graph

$$\begin{tikzcd} & X \arrow[dr, dashed] & \\ S \times \{1, \dots, l\} \arrow[ur, "\prod p_i"] \arrow[dr, "\prod \mathrm{id}"'] & & Y \\ & S \arrow[ur, dashed] & \end{tikzcd}$$

in the category of schemes (over S). Note that $Y = \operatorname{Spec}(B)$, where $B = \{a \in A \mid p_1(a) = \cdots = p_l(a)\}$. Moreover, $\prod p_i : A \to \prod k$ is onto, since the ideals $\ker(p_i)$ of A are comaximal. We thus have a cartesian square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "p"'] & A \arrow[d, "\prod p_i"] \\ k \arrow[r, "\Delta"] & \prod k \end{tikzcd}$$

and an associated Mayer–Vietoris sequence [3, p. 490]. We immediately obtain $\mathrm{SK}_0(Y) \cong \mathrm{SK}_0(X)$ and a sequence

$$0 \to U(Y) \to U(X) \xrightarrow{\prod p_i} \prod U(k)/\Delta U(k) \to \operatorname{Pic}(Y) \to \operatorname{Pic}(X) \to 0.$$

It is clear that $U(Y) = U(X)$ iff X satisfies basepoint independence for units U relative to the points $p_1, \dots, p_l$.

We can extend the Mayer–Vietoris sequence to the left if we replace the Quillen K-theory with the Karoubi–Villamayor K-theory (see Remark 4.2). This is because the regularity of k makes $\prod p_i$ a Gl-fibration [5]. As $K_* = \mathrm{KV}_*$ for the regular schemes S and X [5], the sequence looks like this:

$$\cdots \to K_{n+1}(X) \xrightarrow{\prod p_i^*} \frac{\prod K_{n+1}(S)}{\Delta K_{n+1}(S)} \xrightarrow{\partial} \mathrm{KV}_n(Y) \to K_n(X) \xrightarrow{\prod p_i^*} \cdots.$$

Thus $\mathrm{KV}_n(Y)$ maps onto $K_n(k) \oplus (\bigcap \ker(p_i^*))$. If all the $\ker(p_i^*)$ agree, this is all of $K_n(X)$, so we obtain short exact sequences

$$0 \to (K_{n+1}(S))^{l-1} \xrightarrow{\partial} \mathrm{KV}_n(Y) \to K_n(X) \to 0.$$

Since we are only interested in Quillen's K-theory, we restrict ourselves to $n \leq 2$, where the connection between $K_n(Y)$ and $\mathrm{KV}_n(Y)$ is well known. We first recall the following result of Geller and Roberts [8, Theorem 2], true for any commutative ring k.

Theorem 4.6. *Let $Y = \operatorname{Spec}(B)$ be the scheme obtained from $X = \operatorname{Spec}(A)$ by glueing* *disjoint k-rational points p_i together. Then $\Omega^1_{A/B}=0$ and $K_1(B,I)\cong K_1(A,I)$, where* $I=\ker(p:B\to k)$.

As an easy consequence, we deduce the following result:

Corollary 4.7. *The scheme Y is K_1-regular if X is regular. Hence $K_1(Y)=\mathrm{KV}_1(Y)$ and $\mathrm{NK}_2(Y)\to K_2(Y)\to \mathrm{KV}_2(Y)\to 0$ is exact.*

Proof. The second statement is a well-known consequence of the first [21]. Now Theorem 4.6 for k (and $k[x_1,\dots,x_n]$) give exact rows in the commutative diagram

$$\begin{tikzcd}
K_2(X\times_k \mathbb{A}^n_k) \arrow[r] & \prod_{i=2}^{l} K_2(\mathbb{A}^n_k) \arrow[r] & K_1(Y\times_k \mathbb{A}^n_k) \arrow[r] & K_1(X\times_k \mathbb{A}^n_k) \arrow[r] & \prod_{i=2}^{l} K_1(\mathbb{A}^n_k) \\
K_2(X) \arrow[r] \arrow[u, "="'] & \prod K_2(k) \arrow[r] \arrow[u, "="'] & K_1(Y) \arrow[r] \arrow[u] & K_1(X) \arrow[r] \arrow[u, "="'] & \prod K_1(k). \arrow[u, "="']
\end{tikzcd}$$

The five-lemma gives the K_1-regularity of Y. $\square$

Proposition 4.8. *Let A be an algebra of finite type over a field k. Let B be obtained (as in Definition 4.5) by glueing k-rational points $p_1,\dots,p_l$ of $\mathrm{Spec}(A)$. Then B is an algebra of finite type over k (hence noetherian).*

Proof. We have $B=\{a\in A\mid p_1(a)=\cdots=p_l(a)\}$. Every $a\in A$ satisfies $f(t)=\prod(t-p_i(a))-\prod(a-p_i(a))$, a polynomial with coefficients in B, so A is integral over B. By [20, p. 58], B has finite type over k. $\square$

Example 4.9. Let $X=\mathrm{Spec}(\mathbb{R}[t,t^{-1}])$, and let Y be obtained by glueing $t=1$ and $t=\alpha$, $\alpha\neq 0$ or 1. Then $Y=\mathrm{Spec}(B)$, where

$$\begin{aligned} B&=\{f\in\mathbb{R}[t,t^{-1}]\mid f(1)=f(\alpha)\}\\ &=\mathbb{R}[(t-1)(t-\alpha),\, t(t-1)(t-\alpha),\, t^{-1}(t-1)(t-\alpha)]. \end{aligned}$$

A sketch of the real points is given in Fig. 1. The computation of the K-groups falls into three distinct cases:

$\alpha>0$ $\alpha<0$

Fig. 1. $Y(\mathbb{R})$ in Example 4.9.

Case 1: $\alpha=-1$. We have $\mathrm{SK}_0(Y)=0$ (this is always the case for a 1-dimensional ring). One easily verifies that $U(Y)=\mathbb{R}^*\times\mathbb{Z}=\{at^{2n}\mid a\in\mathbb{R}^*, n\in\mathbb{Z}\}$. From the

Mayer–Vietoris sequence of [3, p. 490] we get $\mathrm{Pic}(Y) \cong \mathbb{R}^*_{>0}$ (multiplicative group of positive real numbers). Because of Theorem 4.6, the Mayer–Vietoris sequence of Quillen K-theory extends to the K_2 terms (the proof is as in [12, p. 55]. This yields $\mathrm{SK}_1(Y) \cong K_2(\mathbb{R})/\{-1,-1\} = K_2^+(\mathbb{C})$.

To compute $\mathrm{KV}_2(Y)$, we have the exact sequence

$$K_3(X) \xrightarrow{p_1^* - p_{-1}^*} K_3(\mathbb{R}) \to \mathrm{KV}_2(Y) \to K_2(X) \xrightarrow{p_1^* - p_{-1}^*} K_2(\mathbb{R}).$$

Now $K_2(X) = K_2(\mathbb{R}) \oplus \{\mathbb{R}^*, t\}$, and since $0 = \{\mathbb{R}^*, 1\} = \{\mathbb{R}^*_{>0}, -1\}$, the kernel (at the K_2 level) of $p_1^* - p_{-1}^*$ is $K_2(\mathbb{R}) \oplus \{\mathbb{R}^*_{>0}, t\}$. Thus $\mathrm{KV}_2(Y)$ maps onto $K_2(\mathbb{R}) \oplus \{\mathbb{R}^*_{>0}, t\}$. By (4.7), $K_2(Y)$ maps onto $\mathrm{KV}_2(Y)$. Thus we have a surjection $K_2(Y) \to K_2(\mathbb{R}) \oplus \{\mathbb{R}^*_{>0}, t\}$. This surjection is split, since for $r > 0$ we have $\{r, t\} = \{\sqrt{r}, t^2\}$ with $t^2 \in B$.

Similarly, $K_3(X) = K_3(\mathbb{R}) \oplus K_2(\mathbb{R})$ and the map $p_1^* - p_{-1}^*$ (at the K_3 level) is seen to be $0 \oplus \{\ ,-1\}$. Again, $\{r, s, -1\} = 0$ unless $r, s < 0$ when it is $\{-1,-1,-1\} \neq 0$. Summarizing, we have:

Proposition 4.10. *When* $\alpha = -1$, *for* Y *as in Example* 4.9, $\mathrm{Pic}(Y) = \mathbb{R}^*_{>0}$, $\mathrm{SK}_0(Y) = 0$, $U(Y) = \mathbb{R}^* \oplus \mathbb{Z}$, $\mathrm{SK}_1(Y) = K_2^+(\mathbb{C})$, *and* $K_2(Y) = K_2(\mathbb{R}) \oplus \mathbb{R}^*_{>0} \oplus (?)$. *There is an exact sequence*

$$\mathrm{NK}_2(Y) \to (?) \to K_3(\mathbb{R})/\{-1,-1,-1\} \to 0.$$

Remark. By Corollary 4.8 of [26], it follows that (?) is $K_3(\mathbb{R})/\{-1,-1,-1\} \oplus \mathbb{R}^+$.

Case 2: $\alpha < 0$, $\alpha \neq -1$. Again we have $\mathrm{SK}_0(Y) = 0$, but $U(Y) = \mathbb{R}^*$. The Mayer–Vietoris sequence yields $\mathrm{Pic}(Y) \cong \mathbb{R}^*/\{\alpha^m\} \cong \mathbb{R}/\mathbb{Z}$ since $\alpha < 0$. Also, $\mathrm{SK}_1(Y) \cong K_2(\mathbb{R})/\{\mathbb{R}^*, \alpha\}$. Now $\{-1, \alpha\} = \{-1, -1\}$ and, as in [12, p. 107] and [19], it follows that $\mathrm{SK}_1(Y)$ is a $\mathbb{Q}$-vector space of uncountable dimension.

Case 3: $\alpha > 0$, $\alpha \neq 1$. We have $\mathrm{SK}_0(Y) = 0$, $U(Y) = \mathbb{R}^*$, $\mathrm{Pic}(Y) = \mathbb{R}^*/\{\alpha^m\}$, and $\mathrm{SK}_1(Y) = K_2(\mathbb{R})/\{\mathbb{R}^*, \alpha\}$ as in Case 2. This time, though, we have $\mathrm{Pic}(Y) \cong \mathbb{Z}/2 \oplus \mathbb{R}/\mathbb{Z}$ and $\mathrm{SK}_1(Y) = \mathbb{Z}/2\mathbb{Z} \oplus V$, V a $\mathbb{Q}$-vector space of uncountable dimension. Note that in this case we can map B to the ring of continuous functions on the circle. The resulting maps $\tilde{K}_0(Y) \to \widetilde{KO}(S^1) = \mathbb{Z}/2\mathbb{Z}$ and $SK_1(Y) \to \pi_1(SO) = \mathbb{Z}/2\mathbb{Z}$ given in [12] are easily seen to be onto. The K-theory is noticing the fact that the circle has not been punctured.

Example 4.11. Let $X = \mathrm{Spec}(A)$, $A = \mathbb{R}[x,y]/(x^2 + y^2 - 1)$, and let Y be obtained by identifying the point $p_0 = (x-1, y)$ to $p_\theta = (x - \cos\theta, y - \sin\theta)$, $\theta \neq 0$. A rotation (by $\theta/2$) and translation allow us to check that $Y = \mathrm{Spec}(B)$, $B = \mathbb{R}[x,y]/(z^2 + x^2(x + \cos\theta/2)^2 - x^2)$. Since $\dim(B) = 1$, $\mathrm{SK}_0(Y) = 0$. We claim that $\mathrm{Pic}(Y) \cong \mathbb{R}^*_{>0} \oplus (\mathbb{Z}/2)^2$. To see this we have to show that the sequence (from [3, p. 490])

$$0 \to \mathbb{R}^* \xrightarrow{\partial} \mathrm{Pic}(Y) \to \mathrm{Pic}(X) \to 0$$

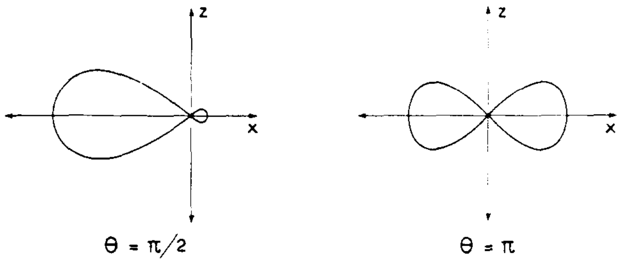

Fig. 2. $Y(\mathbb{R})$ in Example 4.11.

splits. The B-ideals $I_{\pm}=(z,x+\cos(\theta/2)\pm 1)$ both map to the nonzero element of $\mathrm{Pic}(X)=\mathbb{Z}/2$. To show that the sequence splits it suffices to show that one of $\{I_{\pm}\}$ is of order 2. But

$$\begin{aligned}(I_+)^2&=(z^2,z(x+\cos(\theta/2)+1),(x+\cos(\theta/2)+1)^2)B\\&=(x+\cos(\theta/2)+1)B\cdot(x^2(x+\cos(\theta/2)-1),z,x+\cos(\theta/2)+1)B\\&=(x+\cos(\theta/2)+1)B,\end{aligned}$$

since $x+\cos(\theta/2)+1$ and $x^2(x+\cos(\theta/2)-1)$ are comaximal in B. This shows that I_+ has order 2 in $\mathrm{Pic}(B)$. A similar argument works for I_-. In fact it can be shown that $I_-\cong\partial(-1)\otimes I_+$. To compute $\mathrm{SK}_1(Y)$ we utilize the analysis of $K_2(A)$ in Section 2. We have an exact sequence

$$S^1\xrightarrow{p_\theta^*\varphi}K_2(\mathbb{R})\to\mathrm{SK}_1(Y)\to\mathbb{Z}/2\to 0.$$

The left-hand map is $p_\theta^*\varphi:z\to\lambda_*\{z,e^{i\theta}\}$, and evidently cannot map onto the torsion element $\{-1,-1\}$. In any event $\mathrm{coker}(p_\theta^*\varphi)\cong K_2(\mathbb{R})$. Now the splitting for $\mathrm{Pic}(Y)$ gives a splitting here in exactly the same fashion. Thus there is a canonical isomorphism

$$\mathrm{SK}_1(Y)\cong(\mathbb{Z}/2)^2\oplus V,$$

where V is $K_2^+(\mathbb{C})/\lambda^*\lambda_*\{S^1,e^{i\theta}\}$, an uncountable vector space over $\mathbb{Q}$. If $e^{i\theta}$ is a root of unity, we have $V=K_2^+(\mathbb{C})$ since $p_\theta^*\varphi=0$, but this is not so in general.

Proposition 4.12. *When* $\theta=\pi$, $K_2(Y)=K_2(\mathbb{R})\oplus S^1\oplus(?)$, *and there is an exact sequence* $\mathrm{NK}_2(Y)\to(?)\to K_3(\mathbb{R})\to 0$.

Remark. By Corollary 4.8 of [26], it follows that (?) is $K_3(\mathbb{R})\oplus\mathbb{R}^+$.

Proof. We first show that $p_0^*=p_\pi^*:K_3(X)\to K_3(\mathbb{R})$. As in the proof of Proposition 4.4, we have $p_\theta^*\varphi(y)=\lambda_*\{y,e^{i\theta}\}$ for $y\in K_2(\mathbb{C})$. But $\{y,1\}=\{y,-1\}=0$ in $K_3(\mathbb{C})$, so $p_0^*\varphi=p_\pi^*\varphi=0$. By Theorem 2.1, $\mathrm{Im}\,\varphi$ is all of $\tilde{K}_3(A)$. Thus $p_0^*=p_\pi^*$, and the Mayer–Vietoris sequence for KV-theory yields an exact sequence

$$0\to K_3(\mathbb{R})\to\mathrm{KV}_2(Y)\to K_2(X)\to 0.$$

By Corollary 4.7, $K_2(Y)$ maps onto $\mathrm{KV}_2(Y)$, so we have a surjection $K_2(Y) \to K_2(X) = K_2(\mathbb{R}) \oplus S^1$. We now lift $\varphi: S^1 \to K_2(X)$ to $K_2(Y)$ by the map $z \mapsto (B \otimes \lambda)_* \{\sqrt{z}, t^2\}$ (recall that $t^2 \in B$ and $z \in S^1 \subset \mathbb{C}$). Note that z has two square roots, but our lifting is well defined in $K_2(Y)$ because

$$(B \otimes \lambda)_* \{-1, t^2\} = (B \otimes \lambda)_* (B \otimes \lambda)^* \{-1, t^2\} = \{-1, t^2\}^2 = 0.$$

We do have a lifting of φ, since in $K_2(A)$ we have $\lambda_*^A \{\sqrt{z}, t^2\} = \lambda_*^A \{z, t\} = \varphi(z)$. This yields a direct sum decomposition $K_2(Y) = K_2(\mathbb{R}) \oplus S^1 \oplus (?)$; putting this together with Corollary 4.7 yields the exact sequence of Example 4.11. $\square$

We conclude by making a topological remark. The real points of Y look like $S^1 \vee S^1$, so we get maps $\tilde{K}_0(Y) \to KO(S^1 \vee S^1) = (\mathbb{Z}/2)^2$, $\mathrm{SK}_1(Y) \to [S^1 \vee S^1, SO] = \pi_1(SO)^2 = (\mathbb{Z}/2)^2$. By inspection we see that these maps are onto and split by $\eta_\pm = [I_\pm] - 1 \in K_0(Y)$ and $\{\eta_\pm, -1\} \in \mathrm{SK}_1(Y)$.

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